DOI: 10.17725/j.rensit.2024.16.003

Physically realizable reconstruction of a continuous signal after sampling

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Received August 23, 2023, peer-reviewed August 30, 2023, accepted September 07, 2023, published March 15, 2024.

Abstract: **It is proposed to describe signals in a physically realizable basis. Correlation functions of impulse characteristics of physically implemented filters are used as basic functions. To obtain analytical expressions of the functions of the proposed physically realizable basis, it is proposed to use inverse Fourier transforms from approximations (Butterworth, Chebyshev, etc.) of the squares of the amplitude-frequency characteristics of normalized low-pass filters. The basis, functions are copies of the indicated correlation functions of the impulse characteristics, shifted relative to each other by the same time interval, which is the sampling interval. It is shown that there is a sampling theorem in the space of the introduced functions. The exact restoration of the signal is possible if the functions of the considered basis have the property of readability. In this case, the considered basic functions are functions of counts and are not physically implemented. To reduce the error of restoring a continuous signal from its readings using the proposed basis, it is necessary to increase the order of the filter, the pulse characteristics of which are used in the formation of the basis. To restore a continuous signal, its samples must be fed to two cascaded filters. The first filter must have an impulse response, the correlation function of which is used to form a physically realizable basis. The second filter must be matched to the impulse response of the first filter.**

Keywords: **sampling, sampling theorem, pulse characteristic of the filter, correlation function**

UDC 621.376.5

For citation: Andrey N. Degtyarev, Igor L. Afonin, Alexander L. Polyakov. Physically realizable reconstruction of a continuous signal after sampling. *RENSIT: Radioelectronics. Nanosystems. Information Technologies,* 2024, 16(1):3-10e. DOI: 10.17725/j.rensit.2024.16.003.

Contents

1. INTRODUCTION

V.A. Kotelnikov's sampling theorem represents band-limited signals in the series form based on sampling functions. The series coefficients are signal samples taken at times that are multiples of the sampling interval $\alpha = 1/(2f_m)$, where f_m – maximum frequency in the signal spectrum. The sampling functions have the property that when one of them reaches its maximum, the other is equal to zero. To restore a continuous signal from its samples, delta pulses weighted by signal samples are applied to an ideal low-pass filter (LPF). However, ideal LPFs are not physically realized, so Butterworth filters are used as such low-pass filters. The impulse response of Butterworth filters does not have the specified property of sampling functions [1,2,3,13,14,15].

Let's consider the features of discretization and recovery of signals in the basis of physically realizable equidistant functions.

2. MAIN PART

Let the equidistant functions $\varphi_n(t) = \varphi_0(t - n\alpha)$ are orthonormal with weight equal to one. Then some function $f(t)$, which spectral density is equal to

$$F(\mathrm{j}\omega) = F(\omega)\mathrm{e}^{\mathrm{j}\theta_f(\omega)},$$

can be lined up

$$f(t) = \sum_{n=-\infty}^{\infty} y_n \varphi_n(t).$$

Since the basic functions are equidistant $\varphi_n(t)$, then

$$\Phi_n(\mathrm{j}\omega) = \Phi_0(\omega)\mathrm{e}^{\mathrm{j}\theta_\varphi(\omega)}\mathrm{e}^{-\mathrm{j}\omega n\alpha},$$

and the discrete Fourier transform $f(t)$ provides dependence

$$F(\mathrm{j}\omega) = F(\omega)\mathrm{e}^{\mathrm{j}\theta_f(\omega)} = \\ = \Phi_0(\omega)\mathrm{e}^{\mathrm{j}\theta_\varphi(\omega)} \sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{-\mathrm{j}\omega\alpha n}. \quad (1)$$

The mean square of the approximation error $f(t)$ in series is defined as

$$I = \int_{-\infty}^{\infty} [f(t) - \sum_{n=-\infty}^{\infty} y_n \varphi_n(t)]^2 \mathrm{d}t, \quad (2)$$

where

$$y_n = \frac{1}{\|\varphi_n\|^2} \int_{-\infty}^{\infty} f(t)\varphi_n(t)\mathrm{d}t,$$

$$\|\varphi_n\|^2 = E_\phi = \int_{-\infty}^{\infty} \varphi_n^2(t)\mathrm{d}t.$$

Based on the Parseval's identity (2), one can rewrite in the form

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} [F(\mathrm{j}\omega) - \sum_{n=-\infty}^{\infty} y_n \Phi_n(\mathrm{j}\omega)] \times \\ \times [\overset{*}{F}(\mathrm{j}\omega) - \sum_{n=-\infty}^{\infty} y_n \overset{*}{\Phi}_n(\mathrm{j}\omega)]\mathrm{d}\omega = \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} [F^2(\omega) - F(\mathrm{j}\omega)\sum_{n=-\infty}^{\infty} y_n \overset{*}{\Phi}_n(\mathrm{j}\omega) - \\ - \overset{*}{F}(\mathrm{j}\omega)\sum_{n=-\infty}^{\infty} y_n \Phi_n(\mathrm{j}\omega) + \\ + \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} y_n y_k \overset{*}{\Phi}_n(\mathrm{j}\omega)\Phi_k(\mathrm{j}\omega)]\mathrm{d}\omega.$$

One can transform the resulting expression and obtain

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} [F^2(\omega) - F(\omega)\mathrm{e}^{\mathrm{j}\theta_f(\omega)}\Phi_0(\omega)\mathrm{e}^{-\mathrm{j}\theta_\varphi(\omega)}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{\mathrm{j}\omega n\alpha} - \\ - F(\omega)\mathrm{e}^{-\mathrm{j}\theta_f(\omega)}\Phi_0(\omega)\mathrm{e}^{\mathrm{j}\theta_\varphi(\omega)}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{-\mathrm{j}\omega n\alpha} + \\ + \Phi_0^2(\omega)\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} y_n y_k \mathrm{e}^{\mathrm{j}\omega k\alpha}\mathrm{e}^{-\mathrm{j}\omega n\alpha}]\mathrm{d}\omega.$$

Let's consider that $F(\omega) = \Phi_0(\omega)$, and obtain where

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\Phi_0^2(\omega) - \Phi_0^2(\omega)\mathrm{e}^{\mathrm{j}(\theta_f(\omega) - \theta_\varphi(\omega))}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{\mathrm{j}\omega n\alpha} - \\ - \Phi_0^2(\omega)\mathrm{e}^{-\mathrm{j}(\theta_f(\omega) - \theta_\varphi(\omega))}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{-\mathrm{j}\omega n\alpha} + \Phi_0^2(\omega)\sum_{n=-\infty}^{\infty} y_n^2]\mathrm{d}\omega = \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_0^2(\omega)[1 - \mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{\mathrm{j}\omega n\alpha} - \\ - \mathrm{e}^{-\mathrm{j}\Delta\theta(\omega)}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{-\mathrm{j}\omega n\alpha} + \sum_{n=-\infty}^{\infty} y_n^2]\mathrm{d}\omega,$$

where $\Delta\theta(\omega) = \theta_f(\omega) - \theta_\varphi(\omega)$.

Since the coefficients y_n are delta-correlated with each other, one can have

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_0^2(\omega)[2 - \mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{\mathrm{j}\omega n\alpha} - \\ - \mathrm{e}^{-\mathrm{j}\Delta\theta(\omega)}\sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{-\mathrm{j}\omega n\alpha}]\mathrm{d}\omega.$$

Taking into account the evenness of the function $\Phi_0^2(\omega)$ and the oddness of the function $\Delta\theta(\omega)$ as well as the fact that the integration is performed over infinite limits, one can get

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_0^2(\omega)[2-2\mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\sum_{n=-\infty}^{\infty}y_n\mathrm{e}^{\mathrm{j}\omega n\alpha}]\mathrm{d}\omega =$$
$$= 2E_\varphi - 2\sum_{n=-\infty}^{\infty}y_n\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_0^2(\omega)\mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega =$$
$$= 2E_\varphi\left(1-\frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty}y_n\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_0^2(\omega)\mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega\right).$$

The same expression can be obtained when the functions $\varphi_n(t)$ are orthogonal, without considering the coefficients y_n as delta-correlated.

In order for the mean square of the error to be equal to zero, the condition must be satisfied

$$\frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty}y_n\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_0^2(\omega)\mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega = 1,$$

which is possible if

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_0^2(\omega)\mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega = E_\varphi y_n. \quad (3)$$

Consider the coefficient value y_n

$$y_n = \frac{1}{\|\varphi_n\|^2}\int_{-\infty}^{\infty}f(t)\varphi_n(t)\mathrm{d}t = \frac{1}{E_\phi}\int_{-\infty}^{\infty}f(t)\varphi_n(t)\mathrm{d}t.$$

Based on the Parseval's identity, one can write down

$$y_n = \frac{1}{E_\varphi}\int_{-\infty}^{\infty}f(t)\varphi_n(t)\mathrm{d}t =$$
$$= \frac{1}{2\pi}\frac{1}{E_\varphi}\int_{-\infty}^{\infty}\Phi_0(\omega)\mathrm{e}^{\mathrm{j}\theta_f(\omega)}\Phi_0(\omega)\mathrm{e}^{-\mathrm{j}\theta_\varphi(\omega)}\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega = \quad (4)$$
$$= \frac{1}{E_\varphi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_0^2(\omega)\mathrm{e}^{\mathrm{j}\Delta\theta(\omega)}\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega.$$

From comparison (4) and (3) it becomes obvious that equality (3) is an identity. Therefore, the mean square of the signal approximation error (2) equals zero.

Expression (4) can be rewritten as

$$y_n = \frac{1}{2\pi}\frac{1}{E_\varphi}\int_{-\infty}^{\infty}F(\mathrm{j}\omega)\Phi_0(-\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega n\alpha}\mathrm{d}\omega =$$
$$= \frac{1}{E_\varphi}R_{f\varphi}(n\alpha),$$

where $R_{f\varphi}(n\alpha) = \int_{-\infty}^{\infty} f(t)\varphi_0(t-n\alpha)\mathrm{d}t$ – the values of the correlation function of the signal $f(t)$ and the function $\varphi_0(t)$, taken at time points $n\alpha$ (n = 0, 1, 2,...).

In order to determine the value of α, some requirements must be applied to the properties of the signal $f(t)$.

So, if one assumes that the signal $f(t)$ is limited in frequency by the maximum frequency ω_m, then the famous sampling theorem is obtained

Indeed, since the signal is represented as a series

$$f(t) = \sum_{n=-\infty}^{\infty}y_n\varphi_0(t-n\alpha),$$

then its spectral density is equal to (1)

Subsequently, one can write

$$\frac{F(\mathrm{j}\omega)}{\Phi_0(\mathrm{j}\omega)} = \mathrm{e}^{\mathrm{j}[\theta_f(\omega)-\theta_\varphi(\omega)]} = \sum_{n=-\infty}^{\infty}y_n\mathrm{e}^{-\mathrm{j}\omega\alpha n}. \quad (5)$$

The expression $\sum_{n=-\infty}^{\infty}y_n\mathrm{e}^{-\mathrm{j}\omega\alpha n}$ represents a periodic function with a period $2\pi/\alpha$.

Since the ratio of spectral densities $\frac{F(\mathrm{j}\omega)}{\Phi_0(\mathrm{j}\omega)}$ is defined for $\omega\in[-\omega_m,\omega_m]$, then (5) is possible only if

$$\frac{2\pi}{\alpha} = 2\omega_m.$$

where one can obtain

$$\alpha = \frac{2\pi}{2\omega_m} = \frac{\pi}{2\pi f_m} = \frac{1}{2f_m}, \quad (6)$$

which corresponds to the sampling interval introduced by the sampling theorem.

In this case, the signal can be represented in series

$$f(t) = \frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty}R_{f\varphi}\left(\frac{n}{2f_m}\right)\varphi_0\left(t-\frac{n}{2f_m}\right). \quad (7)$$

If the sampling functions are used as equidistant functions

$$\varphi_n(t) = \frac{\sin\omega_m\left(t - \dfrac{n}{2f_m}\right)}{\omega_m\left(t - \dfrac{n}{2f_m}\right)},$$

then $E_\varphi = \pi/\omega_m$, and according to (5) on the interval $\omega \in [-\omega_m, \omega_m]$, one can have

$$F(\mathrm{j}\omega) = \sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{-\mathrm{j}\omega\alpha n}.$$

The coefficients of the series in this case are defined as

$$\int_{-\omega_m}^{\omega_m} F(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega\alpha n}\mathrm{d}\omega = \sum_{n=-\infty}^{\infty} y_n \int_{-\omega_m}^{\omega_m} \mathrm{e}^{-\mathrm{j}\omega\alpha n}\mathrm{e}^{\mathrm{j}\omega\alpha n}\mathrm{d}\omega,$$
$$\int_{-\omega_m}^{\omega_m} F(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega\frac{\pi n}{\omega_m}}\mathrm{d}\omega = y_n \int_{-\omega_m}^{\omega_m} \mathrm{d}\omega. \quad (8)$$

Since

$$f(t) = \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} F(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega, \quad \int_{-\omega_m}^{\omega_m}\mathrm{d}\omega = 2\omega_m,$$

then equality (8) is rewritten in the form

$$2\pi f\left(\frac{\pi n}{\omega_m}\right) = 2\omega_m y_n.$$

Hence one can get

$$y_n = \frac{\pi}{\omega_m} f\left(\frac{\pi n}{\omega_m}\right).$$

Thus, the band-limited signal is expanded into a series of sample functions as follows

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2f_m}\right)\varphi_0\left(t - \frac{n}{2f_m}\right),$$

which corresponds to the sampling theorem.

It is worth noting the sampling theorem is obtained proceeding from the general approximation positions of functions in series in equidistant functions. The assumptions that led directly to the sampling theorem were: firstly, the signal must be limited in frequency and, secondly, the sampling functions are used as equidistant functions [4,5,6,7].

Therefore, the signal expanded in a series of equidistant functions can be written as

$$f(t) = \frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty} R_{f\varphi}(n\alpha)\varphi_0(t - n\alpha).$$

It is rational to determine the dependence of the coefficients in the series on the signal samples.

Let's take the m^{th} sample of this signal

$$\int_{-\infty}^{\infty} f(t)\delta(t - m\alpha)\mathrm{d}t =$$
$$= \frac{1}{E_\varphi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} R_{f\varphi}(n\alpha)\varphi_0(t - n\alpha)\delta(t - m\alpha)\mathrm{d}t.$$

Using the filtering property of the delta function, one can get

$$f(m\alpha) = \frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty} R_{f\varphi}(n\alpha)\varphi_0[(m-n)\alpha]. \quad (9)$$

So, the m^{th} sample of the signal $f(t)$ is calculated through the values of the cross-correlation function of the signal $f(t)$ and the function $\varphi_0(t)$ rather complicated (9), using a discrete filter with an impulse response $\varphi_0(m\alpha)$.

Let's consider what requirements can be imposed on the function $\varphi_0(t)$ to simplify the practical definition of the m^{th} sample of the signal $f(t)$.

Since $R_{f\varphi}(n\alpha)$ are readings of the correlation function of the signal $f(t)$ and the function $\varphi_0(t)$ one can write

$$R_{f\varphi}(n\alpha) = \sum_{k=-\infty}^{\infty} f(k\alpha)\varphi_0[(k-n)\alpha].$$

One can get

$$f(m\alpha) =$$
$$= \frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f(k\alpha)\varphi_0[(k-n)\alpha]\varphi_0[(m-n)\alpha].$$

Let's introduce the replacement $l = k - n$ ($n = k - l$) and one can obtain

$$f(m\alpha) = \frac{1}{E_\varphi}\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f(k\alpha)\varphi_0[l\alpha]\varphi_0[(l+m-k)\alpha] =$$
$$= \frac{1}{E_\varphi}\sum_{k=-\infty}^{\infty} f(k\alpha)\sum_{l=-\infty}^{\infty}\varphi_0[l\alpha]\varphi_0[(l+m-k)\alpha].$$

One takes into account that

$$\sum_{l=-\infty}^{\infty}\varphi_0[l\alpha]\varphi_0[(l+m-k)\alpha]=R_\varphi[(m-k)\alpha],$$

where $R_\varphi[(m-k)\alpha]$ – samples of the correlation function of the elementary signal $\varphi_0(t)$, taken at times $(m-k)\alpha$, and one will get

$$f(m\alpha)=\frac{1}{E_\varphi}\sum_{k=-\infty}^{\infty}f(k\alpha)R_\varphi[(m-k)\alpha]. \quad (10)$$

Let's introduce the replacement $n = m - k$ ($k = m - n$) and one can obtain

$$f(m\alpha)=\frac{1}{E_\varphi}\sum_{n=-\infty}^{\infty}f[(m-n)\alpha]R_\varphi(n\alpha). \quad (11)$$

For this equality to be an identity, it is necessary that for all $n \neq 0$ the readings of the correlation function $R_\varphi(n\alpha) = 0$.

Using the formulated requirement for the correlation function $R_\varphi(n\alpha)$, one can transform the right side of equality (11)

$$\frac{1}{E_\varphi}f(m\alpha)R_\varphi(0)=\frac{1}{E_\varphi}f(m\alpha)E_\varphi=f(m\alpha)$$

and conclude that (11) is an identity.

Take a note that from the point of view of linear transformations, equality (11) is a linear discrete transformation with a reproducing kernel. The transformation kernel is the expression $R_\varphi[(m-k)\alpha]$.

Expression (11) gives reason to believe that the series, which coefficients are signal samples, should be built only using functions that have the character of equidistantly biased correlation functions of the impulse responses of linear systems:

$$f(t)=\frac{1}{E_\varphi}\sum_{k=-\infty}^{\infty}f(k\alpha)R_\varphi(t-k\alpha). \quad (12)$$

Take into account that this expression agrees with the sampling theorem. Indeed, the correlation function of the function

$$\varphi_0(t)=\frac{\sin\omega_m t}{\omega_m t}$$

equals

$$R_\varphi(\tau)=\int_{-\infty}^{\infty}\frac{\sin\omega_m t}{\omega_m t}\frac{\sin\omega_m(t-\tau)}{\omega_m(t-\tau)}\mathrm{d}t=\frac{\sin\omega_m\tau}{\omega_m\tau}. \quad (13)$$

Substituting (13) into (12) leads to the eminent sampling theorem.

Let's expand the signal $f(t)$ in a series of orthogonal functions

$\varphi_n(t) = \varphi_0(t - n\alpha) = R_\varphi(t - n\alpha)$

such that $R_\varphi(0) \neq 0$ and $R_\varphi(n\alpha) = 0$, if $n\alpha \neq 0$.

Let's consider that the modulus of the spectral density of the signal $f(t)$ is equal to the modulus of the spectral density of the functions $R_\varphi(t - n\alpha)$.

One can get

$$f(t)=\sum_{n=-\infty}^{\infty}y_n R_\varphi(t-n\alpha). \quad (14)$$

Let us determine the coefficient of the series y_m

$$y_m=\frac{1}{E_R}\int_{-\infty}^{\infty}f(t)R_\varphi(t-m\alpha)\mathrm{d}t=\frac{1}{E_R}R_{fR}(m\alpha), \quad (15)$$

where $E_R=\int_{-\infty}^{\infty}R_\varphi^2(t-n\alpha)\mathrm{d}t$,

$R_{fR}(m\alpha)$ – m^{th} count of the cross-correlation function of the signal and the function $R_\varphi(t)$.

Expression (14) can be written as

$$f(t)=\frac{1}{E_R}\sum_{n=-\infty}^{\infty}R_{fR}(n\alpha)R_\varphi(t-n\alpha).$$

Let us determine the m^{th} sample of the signal $f(t)$

$$f(m\alpha)=\int_{-\infty}^{\infty}f(t)\delta(t-m\alpha)\mathrm{d}t=$$

$$=\frac{1}{E_R}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}R_{fR}\left(n\alpha\right)R_\varphi\left(t-n\alpha\right)\delta(t-m\alpha)\mathrm{d}t.$$

Using the filtering property of the delta function, one can get

$$f(m\alpha)=\frac{1}{E_R}\sum_{n=-\infty}^{\infty}R_{fR}\left(n\alpha\right)R_\varphi[(m-n)\alpha].$$

Taking into account the specified properties of the function $R_\varphi(t)$, one will obtain

$$f(m\alpha) = \frac{1}{E_R} R_{fR}\left(m\alpha\right) R_\varphi(0) = R_{fR}\left(m\alpha\right).$$

Thus, the coefficients y_m are signal samples $f(t)$, taken at times multiple of α, and a series (14) can be rewritten as

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(n\alpha\right) R_\varphi(t - n\alpha). \quad (16)$$

Let's return to (15). One can rewrite (15) taking into account (14)

$$y_m = \frac{1}{E_R} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_n R_\varphi(t - n\alpha) R_\varphi(t - m\alpha) \mathrm{d}t.$$

Replacing the expression $\tau = t - n\alpha$ and changing the order of summation and integration allows to write down

$$f(m\alpha) = \frac{1}{E_R} \sum_{n=-\infty}^{\infty} y_n \int_{-\infty}^{\infty} R_\varphi\left(\tau\right) R_\varphi\left(\tau + n\alpha - m\alpha\right) \mathrm{d}\tau.$$

Under the integral sign in this equality there is the value of the correlation function $R_{2\varphi}(t)$ of the function $R_\varphi(t)$, determined at the time $t = n\alpha - m\alpha$. One can write down.

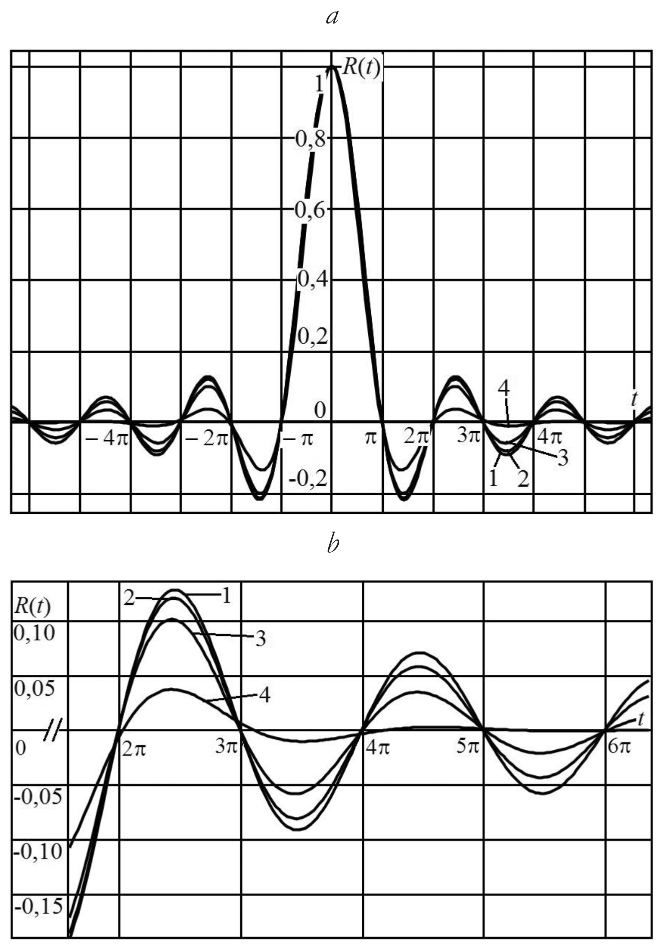
$$f(m\alpha) = \frac{1}{E_R} \sum_{n=-\infty}^{\infty} y_n R_{2\varphi}\left(n\alpha - m\alpha\right).$$

In order to $y_m = f(m\alpha)$, it is necessary to impose conditions on the values of the correlation function $R_{2\varphi}(n\alpha)$, similar to the conditions adopted for the values of the correlation function $R_\varphi(n\alpha)$. That is,

$$R_{2\varphi}(0) \neq 0 \text{ and } R_{2\varphi}(n\alpha) = 0, \text{ if } n\alpha \neq 0.$$

It should be noted that for functions $R_{2\varphi}(t)$ and $R_\varphi(t)$, that can be secured physically (for example, as correlation functions of the impulse responses of linear physically realizable systems), these requirements are met with a certain error (**Fig. 1**). In addition, the system of functions obtained by biasing the correlation function by time intervals multiple of α is not orthogonal. Therefore, series (16) describes the signal approximately.

Thus, in order to approximately restore the signal from its samples, it is necessary to apply samples to two cascade filters, one of which has



1 – function sin*ct*; normalized correlation functions of impulse characteristics of the normalized low-pass Butterworth filter: 2 – 20th order; 3 – 10th order; 4 – 4th order.

Fig. 1. *The function* sin*ct and normalized correlation functions of impulse characteristics of the normalized low-pass Butterworth filter.*

an impulse response $g(t)$, and the second one is consistent with this impulse response, and the correlation function of the impulse response is equal to $R_\varphi(t)$.

The shift interval of equidistant functions, according to which the signal is decomposed (it is the very discretization interval) is chosen based on the given value of the mean square error (2) of the approximation $f(t)$ in series.

In addition, the shift interval of equidistant functions can be chosen taking into account the periodicity of the discrete signal spectrum. The period of the spectrum is related to the upper frequency of the signal spectrum taken into account, i.e. with the upper considered cutoff frequency of the filter which impulse responses are considered [10,11,12].

3. CONCLUSION

In order to approximately restore the signal from its samples, it is necessary to apply samples to two cascade filters, one of which has an impulse response $g(t)$, and the second is consistent with this impulse response. The correlation function of the impulse response must have a correlation function $R_{2\varphi}(t)$, the one that $R_{2\varphi}(0) \neq 0$ and $R_{2\varphi}(n\alpha) = 0$, if $n\alpha \neq 0$.

Since in practice it is impossible to implement filters with similar impulse responses, the signal recovery will occur with some error. This error can be reduced by choosing the appropriate reconstruction filter order and sampling interval.

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