DOI: 10.17725/rensit.2021.13.487

Characteristic form of dynamics equations of Cosserat medium

George G. Bulychev

MIREA-Russian Technological University, https://www.mirea.ru/

Moscow 119454, Russian Federation

E-mail: geo-bulychev@mail.ru

Received August 26, 2021, peer-reviewed September 10, 2021, accepted September 15, 2021

Abstract: **In this paper, we construct the characteristic form of the equations of dynamics of the Cosserat medium and the Cosserat pseudocontinuum for bounded bodies. The method of matrix transformations proposed by the author is used for construction and allows obtaining the necessary relations using identical transformations. The obtained equations are compared with those for a symmetrically elastic isotropic homogeneous body. A method is proposed for selecting the necessary equations for computational schemes at the internal and boundary points of the body. A sequence of operations is proposed for iterative calculations of stresses, particle velocities, moment stresses, and angular velocities of particles in a coupled model of the Cosserat medium.**

Keywords: **dynamic processes, spatial characteristics method, numerical modeling, Cosserat medium**

UDC 537.311.3, 539.23, 546.26

For citation: George G. Bulychev. Characteristic form of dynamics equations of Cosserat medium. *RENSIT: Radioelectronics. Nanosystems. Information technologies*, 2021, 13(4)487-494. DOI: 10.17725/rensit.2021.13.487.

Contents

1. INTRODUCTION

One of the first models of moment elasticity was the brothers Cosserat model [1], developed by them in 1909 (see also [2]). For quite a long time, this model was not in demand and only in the sixties of the last century, applications were found for it, and a significant number of works appeared, one way or another, developed this model [3,4,5]. In particular, in [5] great attention was paid to the kinetics of the medium of Cosserat and, using generating functions, was constructed fundamental solution of the task of elastokinetics and thermoelectrokinetics for this medium. The formulas for the dispersion of the phase velocity of transverse torsion waves were also derived there and the possibility of bifurcation of these waves at different rotational frequencies of medium particles was shown.

The current model Cosserat widely used where the classic theory does not give good agreement with experiment. Thus, when investigating the theory of brittle fracture of metals, Morozov [6] draws on this mathematical theory to describe stresses near the tip of the crack, Zhilin [7] uses the model Cosserat in the theory of non-classical shells, Erofeev and Kunin [8,9] based on nonlinear theory Cosserat studies solitons in solids and develops a polar theory of media with microstructure. There are still a number of works where moment theories are used too study the movement of grains in ferromagnets, thermodiffusion, complicated models of crystal lattices, etc.

In this paper, a matrix and scalar characteristic form of linear medium dynamics equations is constructed on the basis of the matrix method developed by the author for Cosserat line medium and pseudo-continuum Cosserat and discussed the possibilities and sequence of their numerical implementation.

2. CONSTRUCTION OF CHARACTERISTIC EQUATIONS OF DYNAMICS

Following [5], the equations of dynamics for the Cosserat medium can be written as a system consisting of equations of motion

$$\partial_j p_{ji} + f_i = \rho \partial_t^2 u_i, \quad \in_{ijk} p_{jk} + \partial_j \mu_{ji} + y_i = J \partial_t^2 \Omega_i, \tag{1}$$

equations for non-symmetric deformation tensor and bending-torsion

$$\gamma_{ji} = \partial_j u_i - \in_{kji} \Omega_k, \quad \chi_{ji} = \partial_j \Omega_i, \tag{2}$$

and the defining equations

$$p_{ji} = (\mu + a)\gamma_{ji} + (\mu - a)\gamma_{ij} + \lambda \gamma_{kk}\delta_{ji}, \quad \mu_{ji} = (\varphi + \varepsilon)\chi_{ji} + (\varphi - \varepsilon)\chi_{ij} + \beta \chi_{kk}\delta_{ji}. \tag{3}$$

Here μ, λ are the Lame constants, a,φ,ε,β new elastic constants, p_j asymmetric stresses, γ_{ji} asymmetric deformations, χ_{ji} bending-torsion tensor, u_i displacement, Ω_i rotation angles, f_i internal forces, y_i internal moments, $\in_{ijk}$ Levi-Chevita tensor, δ_{ij} the unit tensor, i,j,k = 1,2,3; $\partial_j = \partial/\partial x_j$, $\partial_t^2 = \partial^2/\partial t^2$, t time, x_j Cartesian coordinates, for repeated Roman indices are summed up here and then.

Note that systems (2) and (3) are constructed in such a way that

$$p_{\alpha\alpha} = \sigma_{\alpha\alpha}, \; p_{\alpha\beta} + p_{\beta\alpha} = 2\sigma_{\alpha\beta}, \quad \gamma_{\alpha\alpha} = \varepsilon_{\alpha\alpha}, \; \gamma_{\alpha\beta} + \gamma_{\beta\alpha} = 2\varepsilon_{\alpha\beta}, \tag{4}$$

where $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$, $\varepsilon_{\alpha\alpha}$, $\varepsilon_{\alpha\beta}$ are the stresses of the classical elasticity theory, α, β, γ, α ≠ β ≠ γ, there is no summation here and further along the echoing Greek indices. The indices α,β,γ = 1,2,3 and form a circular permutation of the numbers 1,2,3, i.e. $\in_{\alpha\beta\gamma} = 1$, and $\in_{\beta\alpha\gamma} = -1$.

Remark (4) allows us to reduce some gradient problems of Cosserat medium dynamics (for example, the problem of bound thermoelasticity) to problems of classical momentless mechanics, if, of course, we do not consider asymmetric tangent deformations and stresses separately. Another consequence (4) is that the normal stresses and deformations of the Cosserat medium are equal to the corresponding stresses and deformations of the classical elastic theory, and the interaction of ordinary and moment stresses occurs only through tangent stresses.

Let's proceed to the construction of the matrix form of equations (1)-(3), for which we will enter the row matrices

$$\begin{gathered} P = \|p_{11} \; p_{22} \; p_{33} \; p_{12} \; p_{13} \; p_{21} \; p_{23} \; p_{31} \; p_{32}\|, \\ \Gamma = \|\gamma_{11} \; \gamma_{22} \; \gamma_{33} \; \gamma_{12} \; \gamma_{13} \; \gamma_{21} \; \gamma_{23} \; \gamma_{31} \; \gamma_{32}\|, \\ M = \|\mu_{11} \; \mu_{22} \; \mu_{33} \; \mu_{12} \; \mu_{13} \; \mu_{21} \; \mu_{23} \; \mu_{31} \; \mu_{32}\|, \\ X = \|\chi_{11} \; \chi_{22} \; \chi_{33} \; \chi_{12} \; \chi_{13} \; \chi_{21} \; \chi_{23} \; \chi_{31} \; \chi_{32}\|, \\ U = \|u_1 \; u_2 \; u_3\|, \Omega = \|\Omega_1 \; \Omega_2 \; \Omega_3\|, \\ F = \|f_1 \; f_2 \; f_3\|, Y = \|y_1 \; y_2 \; y_3\| \end{gathered} \tag{5}$$

and two additional matrices-rows

$e_i = \|\delta_1 \; \delta_2 \; \delta_3\|$ and

$$q_{ij} = \|\delta_{1i}\delta_{1j} \; \delta_{2i}\delta_{2j} \; \delta_{3i}\delta_{3j} \; \delta_{1i}\delta_{2j} \; \delta_{1i}\delta_{3j} \; \delta_{2i}\delta_{1j} \; \delta_{2i}\delta_{3j} \; \delta_{3i}\delta_{1j} \; \delta_{3i}\delta_{2j}\|,$$

with which you can select components from matrix rows (5) and restore these matrix rows by their components, for example,

$$e_\alpha U^T = u_\alpha, \; e_i^T u_i = U^T, \; q_{\alpha\beta} P^T = p_{\alpha\beta}, \; q_{ij}^T p_{ij} = P^T, \tag{6}$$

obviously, $e_\alpha e_\beta^T = \delta_{\alpha\beta}$, $e_i^T e_i = I_3$, $q_{\alpha\beta} q_{cd}^T = \delta_{\alpha c}\delta_{\beta d}$, $q_{ij}^T q_{ij} = I_9$, where I_3 and I_9 are unit matrices of 3-rd and 9-th order.

Using matrices (5) and auxiliary matrices e_i and q_{ij} construct the matrix form of equations (1) - (3), using the form of row matrices (5), additional matrices, and properties (6) of additional matrices.

In this case, equations (1) will take the form:

$$Q_i \partial_i P^T + F^T = \rho \partial_t V^T, \quad -S_i Q_i P^T + Q_i \partial_i M^T + Y^T = J \partial_t \omega^T, \tag{7}$$

where $V^T = \partial_t U^T$, $\omega^T = \partial_t \Omega^T$, matrices $Q_i = e_j^T q_{ij}$ and $S_i = - \in_{ijk} e_j^T e_i$ are known to us by [10] and characterize the matrix form of differential invariants *grad, div* and *rot*. In matrix form Q_i and S_i have the form

$$Q_i = \begin{Vmatrix} \delta_{1i} & 0 & 0 & 0 & 0 & \delta_{2i} & 0 & \delta_{3i} & 0 \\ 0 & \delta_{2i} & 0 & \delta_{1i} & 0 & 0 & 0 & 0 & \delta_{3i} \\ 0 & 0 & \delta_{3i} & 0 & \delta_{1i} & 0 & \delta_{2i} & 0 & 0 \end{Vmatrix};$$

$$S_i = \begin{Vmatrix} 0 & -\delta_{3i} & \delta_{2i} \\ \delta_{3i} & 0 & -\delta_{1i} \\ -\delta_{2i} & \delta_{1i} & 0 \end{Vmatrix}.$$

Equations (2), differentiated by t, are written in the matrix representation as

$$\partial_t \Gamma^T = Q_i^T (\partial_i V^T + S_i \omega^T), \quad \partial_t X^T = Q_i^T \partial_i \omega^T. \tag{8}$$

Equations (3), also differentiated by t, in the matrix representation will take the form

$$\partial_t P^T = \{(\mu+a)\, I_9 + (\mu-a) q_{ij}^T q_{ji} + \lambda q_{ij}^T q_{kk} \delta_{ij}\} \partial_t \Gamma^T,$$
$$\partial_t M^T = \{(\varphi+\varepsilon)\, I_9 + (\varphi-\varepsilon) q_{ij}^T q_{ji} + \beta q_{ij}^T q_{kk} \delta_{ij}\} \partial_t X^T, \quad (9)$$

or using (8), finally

$$\partial_t P^T = \{(\mu+a)\, I_9 + (\mu-a) q_{ij}^T q_{ji} + \lambda q_{ij}^T q_{kk} \delta_{ij}\} \times$$
$$\times Q_i^T (\partial_i V^T + S_i \omega^T), \quad (10)$$
$$\partial_t M^T = \{(\varphi+\varepsilon)\, I_9 + (\varphi-\varepsilon) q_{ij}^T q_{ji} + \beta q_{ij}^T q_{kk} \delta_{ij}\} Q_i^T \partial_i \omega^T.$$

Equations (7) and (10), written in the matrix representation, define 24 variables, defined by matrix strings *P, V, M* and ω.

Let's consider an arbitrary element $q_{\alpha\beta}^T q_{\chi\phi} a$, that stands in curly brackets. By selecting the type of matrix rows q_{ij},where $q_{\alpha\beta}^T$ - the column uses its unit to shows the number of the row where the element *a* is located, and - $q_{\chi\phi}$ - the string whose unit shows the sequential number of this element in the row. Noticing this, we write down the stiffness C_p and rotation matrices C_m.

$$C_p = \left\| \begin{matrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu+a & 0 & \mu-a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu+a & 0 & 0 & \mu-a & 0 \\ 0 & 0 & 0 & \mu-a & 0 & \mu+a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu+a & 0 & \mu-a \\ 0 & 0 & 0 & 0 & \mu-a & 0 & 0 & \mu+a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu-a & 0 & \mu+a \end{matrix} \right\|,$$

$$C_m = \left\| \begin{matrix} 2\varphi+\beta & \beta & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 2\varphi+\beta & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \beta & 2\varphi+\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi+\varepsilon & 0 & \varphi-\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi+\varepsilon & 0 & 0 & \varphi-\varepsilon & 0 \\ 0 & 0 & 0 & \varphi-\varepsilon & 0 & \varphi+\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varphi+\varepsilon & 0 & \varphi-\varepsilon \\ 0 & 0 & 0 & 0 & \varphi-\varepsilon & 0 & 0 & \varphi+\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varphi-\varepsilon & 0 & \varphi+\varepsilon \end{matrix} \right\|.$$

Using the constructed matrices C_p and C_m equations (8) and (10) will be written in the form that will be used in the future, i.e. in the form

$$\partial_t P^T = C_p Q_i^T (\partial_i V^T + S_i \omega^T),$$
$$\partial_t M^T = C_m Q_i^T \partial_i \omega^T. \quad (11)$$

Let us construct a matrix form of characteristic equations for stress waves propagating both sides along x_α axis the Cartesian coordinate system $\{x_i\}$. To do this, we multiply the first equation (11) on the left by the matrix Q_α that allocates from the matrix *P* the stresses acting on the site with the normal to the axis x_α, i.e. $Q_\alpha P^T = \| p_{\alpha 1} \quad p_{\alpha 2} \quad p_{\alpha 3} \|^T$, and we draw the necessary grouping of the terms of this equation:

$$\partial_t P_\alpha^T - C_{\alpha\alpha} \partial_t V^T = C_{\alpha k} (\partial_k V^T + S_k \omega^T) +$$
$$+ C_{\alpha\alpha} S_\alpha \omega^T, \quad k = \beta, \gamma, \quad (12)$$

where $C_{\alpha\beta} = Q_\alpha C_p Q_\beta^T$, example, repeated indices *k* is carried out by summation.

Note, that all matrices $C_{\alpha\alpha}$ = *diag*(...), α = 1,2,3 and, they have corresponding values 2μ + λ and μ + α. Denote by $|D_{\alpha p}| = \sqrt{C_{\alpha\alpha} / \rho}$ the modulus of the velocity matrix of stress waves propagating along the axis x_α.

We also regroup the first of the equations of motion (7)

$$\partial_\alpha P_\alpha^T - \rho\, \partial_t V^T = -\partial_k P_k^T - F^T, \quad P_k = Q_k P^T. \quad (13)$$

Then multiply (13) from the left by $|D_{\alpha p}|$ and add (12) and (13). We perform a group, selecting the matrix characteristic operator and assuming that the waves propagate in both directions along the axis x_α, we obtain the matrix characteristic equations.

$$(I_3 \partial_t \pm D_{\alpha p} \partial_\alpha)(P_\alpha^T \mp \rho D_{\alpha p} V^T) =$$
$$= C_{\alpha k} (\partial_k V^T + S_k \omega^T) + C_{\alpha\alpha} S_\alpha \omega^T \mp \quad (14)$$
$$\mp D_{\alpha p} (\partial_k P_k^T + F^T), \quad k = \beta, \gamma.$$

Characteristic equations (14) is easy to fall into 6 scalar equations. Chosen form of notation shows that, due to the unsymmetricity of the matrix S_i, the scalar equations for normal stresses do not contain ω_i, which is a control of the correctness of calculations.

Before proceeding to scalar equations, we note that the calculation $C_{\alpha\beta}$ of matrices is quite simple: first, three rows are selected from the matrix C_p, corresponding to the position of the units in the matrix Q_α. Then, from the rectangular matrix $Q_\alpha C_p$ obtained in this way, the corresponding to the units in the matrix Q_β are selected, and the result is a matrix $C_{\alpha\beta}$. Due to the symmetry of the matrix C_p, it is also easy to show that $C_{\beta\alpha} = C_{\alpha\beta}^T$, the latter greatly simplifies further calculations. The scalar equations corresponding to (14) have the form.

$$(\partial_t \pm c_1 \partial_\alpha)(p_{\alpha\alpha} \mp \rho c_1 V_\alpha) =$$
$$= \lambda(\partial_\beta V_\beta + \partial_\gamma V_\gamma) \mp c_1 (\partial_\beta p_{\beta\alpha} + \partial_\gamma p_{\gamma\alpha} + f_\alpha), \quad (15)$$

$$(\partial_t \pm c_2 \partial_\alpha)(p_{\alpha\beta} \mp \rho c_2 V_\beta) = (\mu + a)\partial_\beta V_\alpha -$$
$$-2a\omega_\gamma \mp c_2 (\partial_\beta p_{\beta\beta} + \partial_\gamma p_{\gamma\beta} + f_\beta), \quad (16)$$

$$(\partial_t \pm c_2\partial_\alpha)(p_{\alpha\gamma} \mp \rho c_2 V_\lambda) = (\mu + a)\partial_\gamma V_\alpha + \\ +2a\omega_\beta \mp c_2(\partial_\beta p_{\beta\gamma} + \partial_\gamma p_{\gamma\gamma} + f_\gamma), \quad (17)$$

where $c_1 \equiv c_{\|} = \sqrt{(2\mu+\lambda)/\rho}$ – is the velocity of longitudinal stress waves falling with a similar velocity in an isotropic body, $c_2 = \sqrt{(\mu+a)/\rho}$ – the velocity of transverse stress waves in the model of Cosserat, V_i – components of the particle velocity matrix V.

To construct the characteristics on fixed discontinuities, we will use equations (11) and (12). From equation (12), we select $\partial_\alpha V^T + S_\alpha \omega^T$, and we get

$$\partial_\alpha V^T + S_\alpha \omega^T = C_{\alpha\alpha}^{-1}\left[\partial_t P_\alpha^T - C_{\alpha k}(\partial_k V^T + S_k\omega^T)\right]. \quad (18)$$

In order, to obtain equations for P_β^T and P_γ^T, we multiply the first equation (11) sequentially on Q_β and Q_γ; we get

$$\partial_t P_\beta^T = C_{\beta k}(\partial_k V^T + S_k\omega^T) + C_{\beta\alpha}(\partial_\alpha V^T + S_\alpha\omega^T), \\ \partial_t P_\gamma^T = C_{\gamma k}(\partial_k V^T + S_k\omega^T) + C_{\gamma\alpha}(\partial_\alpha V^T + S_\alpha\omega^T), \quad (19)$$

and substituting (18) in (19) and making the grouping, we finally obtain the matrix characteristic form of the equations on fixed discontinuities

$$\partial_t(P_\beta^T - C_{\beta\alpha}C_{\alpha\alpha}^{-1}P_\alpha^T) = (C_{\beta k} - C_{\beta\alpha}C_{\alpha\alpha}^{-1}C_{\alpha k})(\partial_k V^T + S_k\omega^T), \\ \partial_t(P_\gamma^T - C_{\gamma\alpha}C_{\alpha\alpha}^{-1}P_\alpha^T) = (C_{\gamma k} - C_{\gamma\alpha}C_{\alpha\alpha}^{-1}C_{\alpha k})(\partial_k V^T + S_k\omega^T). \quad (20)$$

Scalar form of equations (20) it consists of six equations

$$\partial_t(p_{\beta\alpha} - \eta p_{\alpha\beta}) = 4a(\mu+a)^{-1}(\mu\partial_\beta V_\alpha + a\omega_\gamma), \\ \partial_t(p_{\beta\beta} - \nu_1 p_{\alpha\alpha}) = 2\mu[(1+\nu_1)\partial_\beta V_\beta + \nu_1\partial_\gamma V_\gamma], \\ \partial_t p_{\beta\gamma} = \mu(\partial_\beta V_\gamma + \partial_\gamma V_\beta) + a(\partial_\beta V_\gamma - \partial_\gamma V_\beta - 2\omega_\alpha). \\ \partial_t(p_{\gamma\alpha} - \eta p_{\alpha\gamma}) = 4a(\mu+a)^{-1}(\mu\partial_\gamma V_\alpha + a\omega_\beta), \\ \partial_t p_{\gamma\beta} = \mu(\partial_\gamma V_\beta + \partial_\beta V_\gamma) + a(\partial_\gamma V_\beta - \partial_\beta V_\gamma + 2\omega_\alpha), \\ \partial_t(p_{\gamma\gamma} - \nu_1 p_{\alpha\alpha}) = 2\mu[\nu_1\partial_\beta V_\beta + (1+\nu_1)\partial_\gamma V_\gamma]. \quad (21)$$

where $\eta = (\mu - a)/(\mu + a)$, $\nu_1 = \lambda/(2\mu+\lambda)$.

To control the calculations, note that all the equations for normal stress components do not depend on ω_i, and add the third equation (21) with the fifth equation, we obtain the equation known from classical linear elasticity.

$$\partial_t(p_{\beta\gamma} + p_{\gamma\beta}) = 2\mu(\partial_\gamma V_\beta + \partial_\beta V_\gamma). \quad (22)$$

Equations (15)-(17) and (21) form a system of 12 scalar characteristic equations, for the 12 components of matrix-strings P and V, and, given an external vector ω, can be used to determine them.

By α sequentially assigning the values 1,2,3, we obtain 36 scalar equations, of which, in numerical simulation, we must select 12.

As it was theoretically proved in [11] for two-dimensional problems of dynamics of a solid body and practically verified for three-dimensional problems by the author, in order to ensure the stability of the count at the inner point of the medium, it is necessary to choose equations on moving discontinuities for normal stresses $p_{\alpha\alpha}$ and particle velocities V_α. The other equations are selected from the context of the problem, usually they are equations on fixed discontinuities. At the boundary point of the medium, according to [12], boundary conditions, equations on discontinuities moving normally to the boundary under consideration, and equations on fixed discontinuities are used.

Considering equations (15)-(17) and (21), you can notice that:

1. In all equations ω_i, it is included in combinations $a\omega$, and for small a or small ω_i it can be ignored and, thus, the connection between the equations for ordinary and moment stresses can be broken. However, if in the first case we arrive at the classical equations of the dynamics of an symmetrically elastic isotropic body, then in the second case the equations correspond to an unsymmetrically elastic body.
2. Equations of the form (22) can be used to reduce the number of equations connecting ordinary and moment stresses, as a result, the number of equations that includes ω_i can be reduced to three.
3. All equations for normal stresses $p_{\alpha\alpha}$ do not depend on ω_i and coincide with the equations of classical elasticity.
4. From the matrix positivity conditions C_p the value must be in the range from 0 to μ. Hence, the velocity of transverse waves in the Cosserat model is always lower than the velocity of longitudinal waves in an isotropic body.

That is, the longitudinal wave always spreads first, and then cross waves. This speed ratio is necessary and sufficient for the stability of numerical schemes.

Let us now construct a matrix characteristic form of moment stresses. The initial equations are the second equations of systems (7) and (11), i.e. the system of equations

$$-S_iQ_iP^T + Q_i\partial_i M^T + Y^T = J\partial_t\omega^T, \quad \partial_t M^T = C_m Q_i^T \partial_i\omega^T. \tag{23}$$

Note that by changing $-S_iQ_iP^T + Y^T$ to F^T, M^T on P^T, J on ρ and ω^T on V^T we come to the first equation (7), and by replacing C_m on C_p with and adding $S_i\omega^T$ to we come to the first equation (11). Therefore, reasoning in the same way as mentioned above, enter the matrix $M_\alpha^T = Q_\alpha M^T$, $C_{\alpha\beta m} = Q_\alpha C_m Q_\beta^T$, $D_{\alpha m} = \sqrt{C_{\alpha\alpha m}/J}$, α,β = 1,2,3. Note that all $C_{\alpha\alpha m}$ and $D_{\alpha m}$ are diagonal.

Then the matrix equation for moment stresses moving along the axis x_α by analogy with (14) takes the form

$$(I_3\partial_t \pm D_{\alpha m}\partial_\alpha)(M_\alpha^T \mp JD_{\alpha m}\omega^T) = = C_{\alpha km}\partial_k\omega^T \mp D_{\alpha m}(\partial_k M_k^T - S_iQ_iP^T + Y^T). \tag{24}$$

where i = α,β,γ, k = β,γ.

The equations on fixed discontinuities similar to (19) will take the form

$$\begin{aligned}&\partial_t(M_\beta^T - C_{\beta\alpha m}C_{\alpha\alpha m}^{-1}M_\alpha^T) = \\ &= (C_{\beta km} - C_{\beta\alpha m}C_{\alpha\alpha m}^{-1}C_{\alpha km})\partial_k\omega^T, \\ &\partial_t(M_\gamma^T - C_{\gamma\alpha m}C_{\alpha\alpha m}^{-1}M_1^T) = \\ &= (C_{\gamma km} - C_{\gamma\alpha m}C_{\alpha\alpha m}^{-1}C_{\alpha km})\partial_k\omega^T.\end{aligned} \tag{25}$$

In scalar form, equations (24) and (25) represent 9 equations

$$\begin{aligned}&(\partial_t \pm c_3\partial_\alpha)(\mu_{\alpha\alpha} \mp Jc_3\omega_\alpha) = \beta(\partial_\beta\omega_\beta + \partial_\gamma\omega_\gamma) \mp \\ &\mp c_3(\partial_\beta\mu_{\beta\alpha} + \partial_\gamma\mu_{\gamma\alpha} + p_{\gamma\beta} - p_{\beta\gamma} + y_\alpha), \\ &c_3 = \sqrt{(2\varphi+\beta)/J}, \\ &(\partial_t \pm c_4\partial_\alpha)(\mu_{\alpha\beta} \mp Jc_4\omega_\beta) = (\varphi-\varepsilon)\partial_\beta\omega_\alpha \mp \\ &\mp c_4(\partial_\beta\mu_{\beta\beta} + \partial_\gamma\mu_{\gamma\beta} + p_{\alpha\gamma} - p_{\gamma\alpha} + y_\beta), \\ &c_4 = \sqrt{(\varphi+\varepsilon)/J}, \\ &(\partial_t \pm c_4\partial_\alpha)(\mu_{\alpha\gamma} \mp Jc_4\omega_\gamma) = (\varphi-\varepsilon)\partial_\gamma\omega_\alpha \mp \\ &\mp c_4(\partial_\beta\mu_{\beta\gamma} + \partial_\gamma\mu_{\gamma\gamma} + p_{\beta\alpha} - p_{\alpha\beta} + y_\gamma), \\ &\partial_t(\mu_{\beta\alpha} - \eta_m\mu_{12}) = 4\varepsilon\varphi(\varphi+\varepsilon)^{-1}\partial_\beta\omega_\alpha, \\ &\partial_t(\mu_{\beta\beta} - \nu_{1m}\mu_{\alpha\alpha}) = 2\varphi[(1+\nu_{1m})\partial_\beta\omega_\beta + \nu_{1m}\partial_\gamma\omega_\gamma], \\ &\partial_t\mu_{\beta\gamma} = \varphi(\partial_\beta\omega_\gamma + \partial_\gamma\omega_\beta) + \varepsilon(\partial_\beta\omega_\gamma - \partial_\gamma\omega_\beta), \\ &\partial_t(\mu_{\gamma\alpha} - \eta_m\mu_{\alpha\gamma}) = 4\varepsilon\varphi(\varphi+\varepsilon)^{-1}\partial_\gamma\omega_\alpha, \\ &\partial_t\mu_{\gamma\beta} = \varphi(\partial_\gamma\omega_\beta + \partial_\beta\omega_\gamma) + \varepsilon(\partial_\gamma\omega_\beta - \partial_\beta\omega_\gamma), \\ &\partial_t(\mu_{\gamma\gamma} - \nu_{1m}\mu_{\alpha\alpha}) = 2\varphi[\nu_{1m}\partial_\beta\omega_\beta + (1+\nu_{1m})\partial_\beta\omega_\beta],\end{aligned} \tag{26}$$

where $\eta_m = (\varphi-\varepsilon)/(\varphi+\varepsilon)$, $\nu_{1m} = \beta/(2\varphi+\beta)$..

Just as in the case of ordinary stresses, the relations for moment stresses are valid

$$\partial_t(\mu_{\beta\gamma} + \mu_{\gamma\beta}) = 2\varphi(\partial_\beta\omega_\gamma + \partial_\gamma\omega_\beta). \tag{27}$$

Matrix characteristic equations (14), (20), (24) and (25) fully determine the dynamics of the Cosserat medium. They are redundant and their choice is determined by the context of the task. In the numerical solution, the equations for longitudinal waves are essential for implementation. In this case, the computational schemes are stable. In the case of waves of one direction, the matrix characteristic equations give as many scalar equations as the defined variables. In both cases, the computational process is recurrent. Equations (22) and (27), as well as the fact that the normal stresses do not depend on ω_i, greatly simplify the computational process by reducing the number of scalar equations in which p_{ij} and ω_i they are linked.

A significant simplification of the equations occurs in the case of the condition

$$\Omega^T = \frac{1}{2} rot\, u, \tag{28}$$

used for a pseudo-continuum Cosserat.

Recording the operation *rot* in matrix form [10] and using (8), we write

$$\partial_t\Gamma^T = Q_i^T(\partial_i V^T + S_i\omega^T) = = Q_i^T(\partial_i V^T + \frac{1}{2}S_iS_m\partial_m V^T), \tag{29}$$

and the characteristic equations for ordinary stresses are not related to the analogous equations for moment stresses.

After calculating (29),we get the matrix-string $\partial_t\Gamma$ as

$$\partial_t\Gamma = \|\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad \varepsilon_{12} \quad \varepsilon_{13} \quad \varepsilon_{21} \quad \varepsilon_{23} \quad \varepsilon_{31} \quad \varepsilon_{32}\|, \tag{30}$$

where $\varepsilon_{ij} = 2^{-1}(\partial_i V_j + \partial_j V_i)$, $i, j = 1,2,3$ – rates of deformations of the classical elasticity theory.

By entering additional matrices q_{ij} and e_i from formula (5), the method used in constructing the matrix C_p, and by grouping the corresponding terms around operators ∂_i, can be obtained $\partial_i\Gamma^T$ as

$$\partial_t\Gamma^T = \Theta_i^T\partial_i V^T, \tag{31}$$

where

$$\Theta_i = \begin{Vmatrix} \delta_{1i} & 0 & 0 & \delta_{2i}/2 & \delta_{3i}/2 & \delta_{2i}/2 & 0 & \delta_{3i}/2 & 0 \\ 0 & \delta_{2i} & 0 & \delta_{1i}/2 & 0 & \delta_{1i}/2 & \delta_{3i}/2 & 0 & \delta_{3i}/2 \\ 0 & 0 & \delta_{3i} & 0 & \delta_{1i}/2 & 0 & \delta_{2i}/2 & \delta_{1i}/2 & \delta_{2i}/2 \end{Vmatrix}.$$

The first equation of the system (8) written for the pseudo-continuum Cosserat has the form

$$\partial_t P^T = C_p \Theta_i \partial_i V^T, \quad (32)$$

where from

$\partial_t P = \partial_t \| \sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{21} \quad \sigma_{23} \quad \sigma_{31} \quad \sigma_{32} \|$, where σ_{ij} – tensor of classical theory of elasticity. For such stresses, the conditions $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ are met, and therefore the characteristic equations for ordinary and moment stresses are not related.

To determine the wave velocity in a pseudo-continuum model let 's consider the products of matrices $Q_\alpha C_p \Theta_\alpha^T, \alpha = 1,2,3$. By calculation, we can see that all these products are diagonal and have their own number $2\mu + \lambda$ and μ, and their corresponding wave velocities are $c_1 = \sqrt{(2\mu+\lambda)/\rho}$ and $c_2^* = \sqrt{\mu/\rho}$, the same as for an ordinary isotropic body. The only difference is that here the stiffness matrix has the form C_p.

The first equation of the system (7) and equation (32) can be used to construct a matrix characteristic form of the dynamics equations of the pseudocontinuum Cosserat. For moment stresses, the construction remains the same, with the difference that the terms $S_i Q_i P^T$ should be excluded from the equations on moving discontinuities.

Let us construct a matrix characteristic form of the pseudocontinuum dynamics equations Cosserat's by method, to be discussed above.

Let's assume as before, the wave propagates in both directions along the axis x_α. Let us perform transformations similar to (12)-(14) and obtain a matrix characteristic equation on moving discontinuities

$$\begin{aligned} &\left(I_3 \partial_t \pm D_{\alpha p}^* \partial_\alpha\right)\left(P_\alpha^T \mp \rho D_{\alpha p}^*\right) = \\ &= C_{\alpha k}^* \partial_k V^T \mp D_{\alpha p}^* \left(\partial_k P_k^T + F^T\right), \end{aligned} \quad (33)$$

where $D_{1p}^* = diag(c_1, c_2^*, c_2^*),, \ C_{\alpha k}^* = Q_\alpha C_p \Theta_k^T$, or in scalar form:

$$\begin{aligned} &\left(\partial_t \pm c_1 \partial_\alpha\right)\left(\sigma_{\alpha\alpha} \mp \rho c_1 V_\alpha\right) = \lambda\left(\partial_\beta V_\beta + \partial_\gamma V_\gamma\right) \mp \\ &\mp c_1 \left(\partial_\beta \sigma_{\beta\alpha} + \partial_\gamma \sigma_{\gamma\alpha} + f_\alpha\right), \\ &\left(\partial_t \pm c_2^* \partial_\alpha\right)\left(\sigma_{\alpha\beta} \mp \rho c_2^* V_\beta\right) = \mu \partial_\beta V_\alpha \mp \\ &\mp c_2^* \left(\partial_\beta \sigma_{\beta\beta} + \partial_\gamma \sigma_{\gamma\beta} + f_\beta\right), \\ &\left(\partial_t \pm c_2^* \partial_\alpha\right)\left(\sigma_{\alpha\gamma} \mp \rho c_2^* V_\gamma\right) = \mu \partial_\gamma V_\alpha \mp \\ &\mp c_2^* \left(\partial_\beta \sigma_{\beta\gamma} + \partial_\gamma \sigma_{\gamma\gamma} + f_\gamma\right). \end{aligned} \quad (34)$$

On fixed discontinuities, due to the symmetry of tangent stresses, it remains to determine the three stresses $p_{\beta\beta}$, $p_{\gamma\gamma}$ and $p_{\beta\gamma}$. Enter a matrix N that selects from matrix-string P the above mentioned voltages. Obviously, this matrix must have the form $N = e_1^T q_{\beta\beta} + e_2^T q_{\gamma\gamma} + e_3^T q_{\beta\gamma}$. Then we multiply $\partial_t P^T$ from equation (32) on the left by N and select in the right part the term containing ∂_1; denoting $P_N^T = NP^T$ and $C_{Ni}^* = NC_p \Theta_i, i = 1,2,3$ we get the equation

$$\partial_t P_N^T = C_{Nk}^* \partial_k V^T + C_{N\alpha}^* \partial_\alpha V^T, \quad k = \beta, \gamma. \quad (35)$$

Since it is possible to determine all the components of the matrix from formulas (34) P_α, multiply (32) on the left by Q_α and select $\partial_t V^T$ from the resulting equation

$$\partial_\alpha V^T = (C_{\alpha\alpha}^*)^{-1}(\partial_t P_\alpha^T - C_{\alpha k}^* \partial_k V^T). \quad (36)$$

Substituting $\partial_t V^T$ from (36) to (35) and conducting the corresponding grouping, we finally obtain the matrix equation on fixed discontinuities

$$\begin{aligned} &\partial_t \left(P_N^T - C_{N\alpha}^* (C_{\alpha\alpha}^*)^{-1} P_\alpha^T\right) = \\ &= \left(C_{Nk}^* - C_{N1}^* (C_{11}^*)^{-1} C_{1k}^*\right) \partial_k V^T. \end{aligned} \quad (37)$$

The scalar form of this equation has the following form

$$\begin{aligned} &\partial_t \left(\sigma_{\beta\beta} - \nu_1 \sigma_{\alpha\alpha}\right) = \left(1-\nu_1\right)\left[\left(\mu+\lambda\right)\partial_\beta V_\beta + \lambda \partial_\gamma V_\gamma\right], \\ &\partial_t \left(\sigma_{\gamma\gamma} - \nu_1 \sigma_{\alpha\alpha}\right) = \left(1-\nu_1\right)\left[\left(\mu+\lambda\right)\partial_\gamma V_\gamma + \lambda \partial_\beta V_\beta\right], \\ &\partial_t \sigma_{\beta\gamma} = \mu\left(\partial_\beta V_\gamma + \partial_\gamma V_\beta\right). \end{aligned} \quad (38)$$

The characteristic equations (34) and (38) are identical to the analogous equations for an isotropic symmetrically elastic body, which constructed, for example, as a special case in [13].

3. MODELING

The characteristic equations for the Cosserat model constructed here allow us to solve a whole range of problems with connected and unconnected ordinary and instantaneous stresses. In particular,

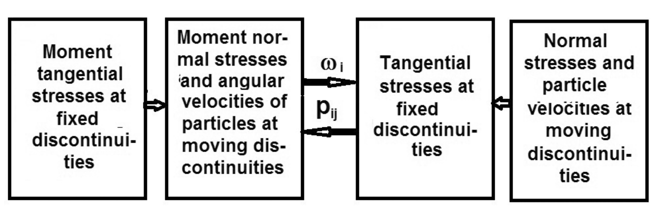

Fig. 1. *Scheme of interaction of longitudinal torsional waves with transverse waves of ordinary stresses and velocities of particles.*

the theory of pseudoconandnouma Kossera allows you to connect classical and moment mechanics and can serve as the first stage of an iterative process of numerical calculation of the related problem of medium dynamicsCosserat.

Consider the case, when $c_1 > c_3 \geq c_2 > c_4$ the point under consideration is also inside the body. In this case, the longitudinal torsion waves act in turn with the transverse waves of ordinary stresses and velocities of a particles. The interaction diagram is shown in **Fig. 1**.

Taking the pseudocontinuum Cosserat as the initial approximation we have unconnected characteristic equations for ordinary and moment stresses. Having determined the angular velocities of particles ω_i from the left part of the Fig. 1, we substitute them in the equations for tangent stresses; then we return the calculated tangent p_{ij} stresses back to the moment equations on moving discontinuities, thus providing a recurrent calculation ω_i and p_{ij}. At the same time, normal stresses $p_{\alpha\alpha}$, their corresponding velocity V_α and tangent moment stresses $\mu_{\alpha\beta}$ do not directly participate in the exchange, although $\mu_{\alpha\beta}$ they are corrected after each exchange act. To reduce the number of equations involved in the exchange, use equations (22), (25), and (32).

To construct computational schemes at the boundary point of the medium at the same speed sequence, we use the scheme shown in **Fig. 2**.

As indicated above, at the boundary point, we should consider the equations for waves moving inside the body along the normal path to its boundary, equations for fixed discontinuities, and boundary conditions. The sum of these equations and conditions is the same as that of the defined variables, but the sequence of calculations depends significantly on the type of boundary conditions.

Consider the right-hand side of Fig. 2, in which p_{ij} and V_i are calculated. As shown by formulas (15), (21),

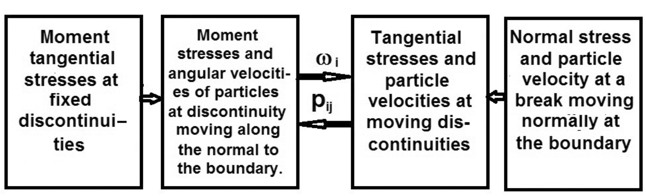

Fig. 2. *Scheme for constructing computational circuits at the boundary point of the medium with the same speed calculation sequence.*

all normal stresses p_{ii} and the velocity of a particles V_α, directed along the normal to the considered boundary x_α, can be calculated independently of ω_i by the same formulas as in the usual metric theory of elasticity. The order of calculation of the remaining values depends on the corresponding boundary conditions: let, for example, tangent voltage $p_{\beta\alpha}$ is given at the boundary x_α, then, in accordance with the equation

$$\partial_t(p_{\alpha\beta} + p_{\beta\alpha}) = 2\mu(\partial_\beta V_\alpha + \partial_\alpha V_\beta) \quad (39)$$

it is determined $p_{\alpha\beta}$, and from equation (16) it is determined V_β, the plus sign or minus is determined by the position of the boundary: plus corresponds to the boundary $x_{\alpha\max}$, and minus – $x_{\alpha\min}$.

If set at the boundary V_β, then from (16) it is defined $p_{\alpha\beta}$, and then from (39) $p_{\beta\alpha}$

Similarly, $p_{\gamma\sigma}$, $p_{\alpha\gamma}$ and V_γ are calculated.

The values $p_{\beta\gamma}$ of and $p_{\gamma\beta}$ are determined from the system (21) and equation (22)

Now consider the left-hand side of Fig. 2, where the angular velocity of particles ω_i and moment stresses μ_{ij} are calculated. We will also start the review with the internal point of the environment. For the stability of the computational scheme, it is necessary to select all the equations for the normal components of moment stresses $\mu_{\alpha\alpha}$, $\alpha = 1,2,3$ and angular velocities of particles ω_α; the remaining stresses can be obtained independently from 3 equations on fixed discontinuities

$$\partial_t \mu_{\alpha\beta} = \varphi(\partial_\alpha \omega_\beta + \partial_\beta \omega_\alpha) + \varepsilon(\partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha), \quad (40)$$

and equations (27). At the boundary point, we define 3 equations on discontinuities moving along the normal to the boundary $x_\alpha = const$ from inside the body, 3 equations on fixed discontinuities, 3 boundary conditions, and 3 equations included in (27). Depending on the boundary condition (ω_i or μ_{ij}) on the moving discontinuities, the remaining variable (μ_{ij} or ω_i) is determined. Then with the help of (27) it is determined μ_{ji} and, finally, the last 3 stresses $\mu_{\beta\beta}$, $\mu_{\gamma\gamma}$ and $\mu_{\beta\gamma}$are divided by the corresponding equations of the system (26).

4. CONCLUSION

In the article, using the matrix apparatus proposed by the author, we construct the matrix and scalar forms of characteristic equations of medium dynamics Cosserat and pseudo-continuum Cosserat. We used the simplest auxiliary matrices consisting of zeros and ones, and the same transformations. The order of derivatives was never increased, and no additional conditions were set. The number of constructed equations turned out to be excessive, so then the question of their choice in various cases was considered.

It was shown that the characteristic form of representation of equations for a pseudo-continuum Cosserat reduced to the equations for symmetrical elastic isotropic body, but with the stiffness matrix of the model Cosserat. This circumstance was used in numerical simulation of the dynamics of the Cosserat medium at the same time pseudo-continuum model was considered as an initial approximation for the scheme of sequential calculations of the dynamics of bounded bodis.

According to [11], equations for longitudinal waves propagating in all three directions were used inside the body. At the boundary points, according to [12], we considered all types of waves that fit normally to the boundary of the body from within. Since in each type of wave that approaches the boundary, the same variables ($p_{\alpha i}$, V_i or $\mu_{\alpha i}$, ω_i) change over time as in the corresponding boundary conditions, the choice of variants of interaction of the wave with the boundary conditions is minimal, and this interaction self it is the simplest and most convenient for numerical calculations.

When approximating the time derivative with a one-way difference and the coordinates with a central difference, the residual terms in all equations have an order $O(\partial_t^2, \partial_{x_i}^3)$ and the calculation errors do not accumulate [13], and the computational schemes are stable.

REFERENCES

1. Cosserat E, Cosserat F. *Theorie des corps deformables.* Paris, Hermann, 1909.
2. Kossera E, Kossera F. Zametka o teorii evklidovskogo deystviya (reprint) [Cosserat E, Cosserat F. A note on the theory of Euclidean action]. *RENSIT,* 2013, 5(1):5-76 (in Russ.); https://www.elibrary.ru/download/elibrary_19429292_61149270.pdf/
3. Kuvshinsky EV, Aero EL. Continuum theory of asymmetric elasticity. *FTT,* 1963, 5(9):2591-2598.
4. Palmov NA. Basic equations of the theory of asymmetric elasticity. *PMM*, 1964, 28(3):401-408.
5. Novatskiy V. *Teoriya uprugosti* [Theory of elasticity]. Moscow, Mir Publ., 1975, p. 852.
6. Morozov NF. *Matematicheskie voprosy teorii theshchin* [Mathematical theory of cracks]. Moscow, Nauka Publ., 1984, p. 256.
7. Zhilin PA. Basic equations of non-classical elasticity theory shells. *Trudy Leningr. polytechn. inst., no. 386*, 1982, pp. 29-46.
8. Erofeev VI. *Volnovye processy v tverdykh telakh s mikrostrukturoy* [Wave processes in solids with microstructure]. Moscow, Moscow state university Publ., 1999, p. 327.
9. Kunin IA. *Teoriya uprugikh tel s mikrostrukturoy. Nelokalnaya teoriya uprugosti* [Theory of elastic media with microstructure. Nonlocal theory of elasticity]. Moscow, Nauka Publ., 1975, p. 415.
10. Bulychev GG. Characteristic form of equations of dynamics of media of complex structure. *RENSIT*, 2020, 12(2):245-252. DOI: 10.17725/rensit.2020.12.245.
11. Kukudzhanov VN. *Chislennoe reshenie neodnomernykh zadach rasprostraneniya voln napryazhenii v tverdykh telakh* [Numerical solution of inhomogeneousproblems of distribution ofstress waves in solids]. Moscow, Dorodnicyn Computing Centre of USSR Publ., Issue 6, 1976, 76 p.
12. Butler DS. The numerical solution of hyperbolic systems of partial differential equations in three independent variables. *Proc. Roy. Soc. of London. Series A, Mathematical and Physical Sciences.* 1962, 255(1281):319-343. DOI: 10.2307/2413911.
13. Samarskiy AA. *Teoriya raznostnykh skhem* [Theory of difference schemes]. Moscow, Nauka Publ., 1989, 616 p.