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Mathematical Foundations of the Fractal Scaling Method in Statistical Radiophysics and Applications

Alexander A. Potapov

Kotelnikov Institute of Radioengineering and Electronics of Russian Academy of Sciences, http://cplire.ru/
Moscow 125009, Russian Federation
Jinan University, Sino-Russian Laboratory of Information Technologies and fractal signal processing, https://english.jnu.edu.cn/
Guangzhou 510632, China
E-mail: potapov@cplire.ru

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Abstract: **The system of basic mathematical concepts and constructions underlying the modern global fractal-scaling method developed by the author is presented. An overview of the main results on the creation of new information technologies based on textures, fractals (multifractals), fractional operators, scaling effects and nonlinear dynamics methods obtained by the author and his students for more than 40 years (from 1979 to the present) at the V.A. Kotelnikov Institute of Radioengineering and Electronics of RAS. It is shown that, for the first time in the world, new dimensional and topological (and not energy!) Features or invariants were proposed and then effectively applied for problems in radio physics and radio electronics, which are combined under the generalized concept of "sample topology" ~ "fractal signature". The author discovered, proposed and substantiated a new type and new method of modern radar, namely, fractal-scaling or scale-invariant radar. It should be noted that fractal radars are, in fact, a necessary intermediate stage on the path of transition to cognitive radar and quantum radar.**

Keywords: **fractal, scaling, fractional operator, texture, non-Markov random process, signature, nonlinear dynamics, radiophysics, radar**

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1. INTRODUCTION

The term "fractal" at the end of the last century was perceived as exotic. Somewhat exaggerating, we can say that fractals formed a thin amalgam on the powerful skeleton of science at the end of the 20th century. The situation has radically changed with the use of fractal structures in technical applications to the processing of stochastic signals and images, artificial intelligence, propagation and scattering of radio waves, electrodynamics, the design of antenna devices, other electrodynamic and radio engineering structures, radioelements with fractal impedance, etc. [17,42,45,57,58,62,73,79, 82,83].

Currently, we can confidently talk about the design of fully fractal radio systems. At the same time, physicists included in their arsenal a new mathematical apparatus, and mathematicians were enriched with new heuristic considerations and joint problem statements.

The purpose of this work is to give, as much as possible, a closed presentation of the basic concepts and mathematical theory for problems and applications of statistical radiophysics, using various approaches to the synthesis of the global fractal-scaling method developed by the author.

The problem presented in the title of the work began to be studied for the first time in the world by the author more than 40 years ago at the Institute of Radioengineering and Electronics (IRE) of the Academy of Sciences of the USSR in connection with the implementation of a cycle of fundamental research devoted to the creation of new breakthrough radiophysical technologies for radar. The main one is detection by a one-dimensional (probabilistic statistical signal) and multidimensional (stochastic optical and radar images) sample of various low-contrast objects against a background of intense interference from the Earth's surface. The research is carried out within the framework of the fundamental scientific direction "Fractal radiophysics and fractal radio electronics: design of fractal radio systems", initiated and developed by the author at the V.A. Kotelnikov IRE of Russian Academy of Sciences (RAS) from 1979 to the present [82,83].

The relevance of these studies is associated with the need for a more accurate description of real processes occurring in radiophysical and radio engineering systems. This is, first of all, taking into account heredity, non-Gaussianity and scaling (self-similarity, self-similarity) of physical signals and fields. All these concepts are included in the description of fractal sets or fractals, first proposed in 1975 by B. Mandelbrot [115].

Naturally, the text does not claim to be complete; detailed evidence is lacking. All the concepts used are introduced along the way. The main purpose of this work is to acquaint the reader with the created texture and fractal (multifractal) methods, as well as their application in general. The reader will find more detailed information and necessary proofs in the author's books and original works on this topic, indicated at the end of this work. Although the choice of material for the review could not but be influenced by the author's mathematical tastes and interests, he hopes that the most fundamental concepts of a fundamental nature are reflected here in sufficient detail.

2. THEORETICAL ASPECTS OF THE METHOD

2.1. Fundamentals of fractional measure and noninteger dimension theory

The main property of fractals is the non-integer value of their dimension. The development of dimension theory began with the work of Poincaré, Lebesgue, Brouwer, Uryson, and Menger. In various areas of mathematics, sets arise in one sense or another that are negligible and indistinguishable in the sense of the Lebesgue measure. To distinguish such sets with a pathologically complex topological structure, it is necessary to use unconventional characteristics of smallness, for example, capacity, potential, Hausdorff measures and dimension, etc. The most fruitful was the use of the Hausdorff fractional dimension, closely related to the concepts of entropy, fractals and strange attractors in the theory of dynamical systems [2,3,8,18,21,24,33,62,77,90,94,95,134].

Fractional Hausdorff dimension is determined by a *p*-dimensional measure with an arbitrary real positive number *p*, which was introduced by Hausdorff in 1919. In the general case, the concept of a measure is not related either to a metric or to a topology. However, the Hausdorff measure can be constructed in an arbitrary metric space on the basis of its metric, and the Hausdorff dimension itself is related to the topological dimension. The concepts introduced by Hausdorff are based on the construction of Carathéodory (1914) [77,105]. Let be (M,ρ) a metric space, F a family of subsets of the space M, and f a function on F such that $0 \le f(G) \le \infty$ for $C \in F$ and $f(\varnothing) = 0$. We construct auxiliary measures m_f^{ε} and then the main measure Λ_f as follows. For $E \subset M$ and $\varepsilon > 0$, the value m_f^{ε} is defined as the exact lower bound of the set of numbers

$$m_f^{\varepsilon} = \inf \sum_i f(\mathrm{G}) \tag{1}$$

over all countable ε-coverings $\{G_i\}$, $G_i \in$ F.

The inequality $m_f^{\varepsilon_1}(E) \ge m_f^{\varepsilon_2}(E)$ for $\varepsilon_2 > \varepsilon_1$ implies the existence of the limit

$$\Lambda(E) = \lim_{\varepsilon \to 0+} m_f^{\varepsilon}(E) = \sup m_f^{\varepsilon}(E). \tag{2}$$

It is clear that m_f^{ε} and $\Lambda(E)$ are also *outer measures* on M. Let $\rho(a,B) > \varepsilon > 0$. Consider an arbitrary ε-covering $\{G_i\}$ of the set $A \bigcup B$, consisting of a certain number of sets. Then the family $\{A \bigcap G_i\}$ and $\{B \bigcap G_i\}$ do not intersect and cover the sets A and B, respectively, therefore

$$m_f^{\varepsilon}(A \bigcup B) \ge \mathrm{m}_f^{\varepsilon}(A) + \mathrm{m}_f^{\varepsilon}(B) \tag{3}$$

or

$$\Lambda_f(A \bigcup B) = \Lambda_f(A) + \Lambda_f(B). \tag{4}$$

The class of Λ_f-measurable sets of the space M form a σ-ring on which the outer measure Λ_f is regular. The measure Λ_f is also called the result of applying the Carathéodory construction to the function f, and the outer measure m_f^{ε} is called the approximating measure of order ε. The measure Λ_f rather subtly reflects the properties of the function f and the family F, although it is usually not an extension of f.

We indicate two simple statements that describe the behavior of approximating measures on a decreasing sequence $C_1 \supset C_2 \supset$ ... of compact subsets of the space M. If the elements of the family F are open subsets of M, then

$$\lim_{i\to\infty} m_f^{\varepsilon}(\mathrm{G}_i) = m_f^{\varepsilon}(\bigcap_{i=1}^{\infty} C_i). \quad (5)$$

If $0 < \varepsilon_0 < \varepsilon$ and $f(S) = \inf\{f(T)\}$: $T \in F$, $S \subset \mathrm{Int}T$, $d(T) \leq \varepsilon$ for all $S \in F$ such that $d(S) \leq \varepsilon$

$$\lim_{i\to\infty} m_f^{\varepsilon}(\mathrm{G}_i) \leq m_f^{\varepsilon}(\bigcap_{i=1}^{\infty} C_i), \quad (6)$$

where d is the diameter of the sets, Int is the set of all interior points of the set T.

Let X be a bounded compact metric space, F be the family of all nonempty compact sets from X, a function f: $F \to [0, +\infty]$ continuous with respect to the Hausdorff metric, and $f(C) > 0$ for all $C \in F$ such that $d(\mathrm{C}) > 0$. If $A_1 \subset A_2 \subset A_3 \subset$ ... form an increasing sequence of subsets of the space X, then

$$\lim_{k\to\infty} m_f^{\varepsilon}(A_k) \leq m_f^{\varepsilon}(\bigcap_{k=1}^{\infty} A_k). \quad (7)$$

Let us define the h-Hausdorff measure. Let $h(r)$ be a continuous monotonically increasing function of r ($r \geq 0$) for which $h(0) = 0$. The class of such functions is denoted by H_0. Applying the Carathéodory construction to the function $f(E) = h[d(E)]$ for $E \neq \emptyset$ and $f(\emptyset) = 0$ (here $d(E)$ is the diameter of the set E), we obtain the Λ_h-Carathéodory measure, which is called h-Hausdorff measure. If, in addition, $h(r) = \gamma(\alpha) r^{\alpha}$, where α is a fixed positive, not necessarily an integer, and $\gamma(\alpha)$ is a positive constant depending only on α, then the h-Hausdorff measure is called an α-dimensional measure or an α-the Hausdorff *measure* H_{α}, which is a Borel regular measure.

The construction of the Hausdorff h-measure can be imagined as follows. Cover α with an arbitrary sequence of disks C_v of radius $r_v \leq \varepsilon$ ($\varepsilon > 0$; $v = 1,2,...$) and denote by $m_h^{\varepsilon}(a,h) \geq 0$ the lower bound of the corresponding sums $\sum_{v=1}^{\infty} h(r_v)$. This number increases with decreasing ε. A-priory

$$\Lambda_h(E) = \lim_{\varepsilon\to 0} m_h^{\varepsilon}(a,h), \quad (8)$$

hence

$$0 \leq \Lambda_h(E) \leq +\infty. \quad (9)$$

Limit (8) is the *outer* h-Hausdorff measure, which is a Borel regular measure on the σ-ring Λ_h of measurable sets of the space M. Choosing different functions for $h(r)$, we get: a linear measure $h(r) = 2\pi r$, a flat measure $h(r) = \pi r$, and a logarithmic measure $h(r) = 1/\ln r$.

Condition $E_1 \subset E_2$ implies $\Lambda_h(E_1) \leq \Lambda_h(E_2)$, that is, the Hausdorff h-measure is a monotonically increasing set function. Using the h-measure, the dimension of the set is defined as follows. If $0 < \Lambda_h(A) < \infty$, then $\langle h \rangle$ is called the metric dimension (Hausdorff dimension) of the set A. If $h(r) = cr^{\alpha}$ and $0 = \Lambda_h(A) < \infty$, then the dimension of the set A is denoted by $\langle \alpha \rangle$, here c is a constant. A set of a certain dimension has an h-measure equal to 0 for each external dimension and ∞ for each lowest h-measure.

A further generalization of the notion of dimension is the *Hausdorff-Besicovitch dimension,* which is introduced through the nonnegative numbers $\alpha_0 = \alpha_0(E)$ in the form of the equality

$$\alpha_0(E) = \sup\{\alpha : H_{\alpha}(E) \neq 0\} = \inf\{\alpha : H_{\alpha}(E) = 0\} \quad (10)$$

for the set E. The dimension of the Hausdorff-Besicovitch set is determined by the behavior of $H_{\alpha}(E)$ not as a function of E, but as a function of α.

The correctness of definition (10) confirms the following property of the H_{α}-measure. If $H_{\alpha}(E) < \infty$, then $H_{\alpha}(E) = 0$ for any $\alpha_2 > \alpha_1$. If the measure $H\alpha_2(E)$ is nonzero, then $H\alpha_1(E) = \infty$ for any positive $\alpha_1 < \alpha_2$. This implies that for the set $E \subset M$ or $H_{\alpha}(E) = 0$ for any $\alpha > 0$, then $\alpha_0(E) = 0$ by definition, or there is a "jump" point α_0 such that $H_{\alpha}(E) = \infty$ for $\alpha < \alpha_0$ and $H_{\alpha}(E) = 0$ for $\alpha > \alpha_0$. This number α_0 is the Hausdorff-Besicovitch dimension.

If, when determining the H_α-Hausdorff measure, the coverings are carried out by balls of the same diameter, then such a measure is called entropy. Then the dimension (10) is called the entropy or Kolmogorov dimension.

For sets of positive k-dimensional Lebesgue measure, both dimensions coincide and are equal to K. The Hausdorff-Besicovitch dimension characterizes an external property of the set. Therefore, it is advisable to introduce the concept of the Hausdorff-Besicovitch set at a point that would characterize its internal structure.

In this case, the number

$$\alpha_E(x_0) = \lim_{n\to\infty} \alpha_0(E \bigcap O_n(x_0)) \tag{11}$$

is called the local Hausdorff-Besicovitch dimension of the set E at the point x_0. Here $\{O_n(x_0)\}$ is an arbitrary sequence of contracting neighborhoods of the point $x_0 \in M$.

Each bounded closed set E of an m-dimensional Euclidean space contains a point $x_0 \in E$ such that

$$\alpha_E(x_0) = \alpha_0(E). \tag{12}$$

A function $\alpha_E(x)$ is called a function of local Hausdorff-Besicovitch dimension if

$$\begin{aligned} &0 \le \alpha_E(x) \le \alpha_0(E) \text{ for any } x \in M, \\ &\alpha_E(x) = 0 \text{ if the set } E \text{ is closed and } x \notin E, \\ &\alpha_E(x) = 0 \text{ for all isolated points of the set } E. \end{aligned} \tag{13}$$

The Hausdorff-Besicovitch dimension is a metric concept, but there is a fundamental connection with the topological dimension dim$E$, which was established by L.S. Pontryagin and L.G. Shnirelman [18, p. 210], introducing in 1932 the notion of metric order, namely: the infimum of the Hausdorff-Besicovitch dimension for all metrics of the compact set E is equal to its topological dimension: dim$E \le \alpha(E)$. One of the widely used methods for estimating the Hausdorff dimension of sets, known as the principle of mass distribution, was proposed by Frostman in 1935 [58,80].

Sets, the Hausdorff-Besicovitch dimension of which is a fractional number, are called *fractal sets* or *fractals*. More strictly, a set E is called a fractal (fractal) in a broad sense (in the sense of B. Mandelbrot) if its topological dimension does not coincide with the Hausdorff-Besicovitch dimension, namely, $\alpha_0(E) >$ dim$E$. For example, the set E of all irrational points [0; 1] is fractal in the broad sense, since $\alpha_0(E) = 1$, dim$E = 0$. The set E is called fractal (fractal) in the narrow sense if $\alpha_0(E)$ is not integer. A fractal set in a narrow sense is the same and in a wide sense.

As was first shown by A.S. Besicovitch in 1929, there are profound differences between Lebesgue sets and fractals. First of all, these features relate to densities. The geometric properties of the fractal set E are determined by the behavior of the function

$$D(\mathrm{x},\varepsilon) = \frac{H_\alpha(E \bigcap O(x,\varepsilon))}{\varepsilon^\alpha} \tag{14}$$

for small ε, where x is an arbitrary point of the set E. The upper α-density of a set E at a point x is called

$$\overline{D}_\alpha(E,x) = \overline{\lim_{\varepsilon\le 0}} D(E,x), \tag{15}$$

respectively, the lower α-density of the set E at the point x is written in the form

$$\underline{D}_\alpha(E,x) = \underline{\lim_{\varepsilon\le 0}} D(x,\varepsilon). \tag{16}$$

When $\overline{D}_\alpha(E,x) = \underline{D}_\alpha(E,x)$, that common value is called the α-density of the set E at the point x and is denoted by $D_\alpha(E, x)$. If $\varepsilon \to 0+$, then $\overline{D}_\alpha(E,x)$ and $\underline{D}_\alpha(E,x)$ it is called right-sided, for $\varepsilon \to 0-$ left-sided, and for $\varepsilon \to 0$, two-sided upper and lower α-density, respectively.

It can be noted that for almost all (in the sense of H_α-Hausdorff) points α of the set on the line, the one-sided upper (right and left) α-density is equal to one, and the one-sided lower α-density is equal to 0 ($0 < \alpha < 1$). For two-sided densities, at almost all points of the α-set on the line, there is no two-sided α-density, that is, the upper α-density is different from the lower one.

2.2. Homogeneous functions and scaling

The last four decades have been a period of significant progress in the physics of fractals and their applied aspects. Experimenters and theorists have successfully used the concept of fractality in the study of numerous physical phenomena. At the same time, the theory of large-scale transformations (the theory of self-similarity, scaling theory) has also received great development. This is most fully reflected in the problem of phase transitions (see, for example, [96,97]).

In fact, in all natural and artificial dynamical systems, it is necessary to take into account the effects of scaling, i.e. the presence of many different spatial and/or temporal scales and all kinds of interactions between them. It is useful to approach the discussion of scaling ideas from the point of view of homogeneous functions.

As follows from [13], a function of one or several variables satisfying the condition that when all arguments of the function $f(x, y, ..., u)$ are simultaneously multiplied by the same arbitrary factor λ, the value of the function is multiplied by some power α of this factor is called homogeneous:

$$f(\lambda x, \lambda y, ..., \lambda u) = \lambda^\alpha f(x, y, ..., u), \tag{17}$$

where α is the order of homogeneity, or the measurement of a homogeneous function.

For example, the power function $f(t) = bt^\alpha$ satisfies the homogeneity relation (17) or *scaling*:

$$f(t) = \lambda^\alpha f(t) \tag{18}$$

for all positive values of the scale factor λ. Naturally, the power function, like many other functions that satisfy the scaling relation (18), are not fractal curves. However, many types of fractals (scale-invariant fractals) have scaling symmetry. Homogeneous functions have many properties that make them very attractive for an approximate description of real processes and objects.

There are: (1) – positively homogeneous functions for which equality (17) holds only for positive $\lambda(\lambda > 0)$, and (2) – absolutely homogeneous functions for which the equality holds:

$$f(\lambda x) = |\lambda|^\alpha f(x) \tag{19}$$

From the differential properties of homogeneous functions, we note *Euler's lemma*: "Homogeneous functions are proportional to the scalar product of their gradient by the vector of their variables with a coefficient equal to the order of homogeneity:

$$\vec{x} \cdot \nabla f(\vec{x}) = \alpha f(\vec{x})." \tag{20}$$

In [13], a specially normalized power function was introduced

$$f_\lambda(t) = \frac{1}{\Gamma(\lambda+1)} t^\lambda, \ t > 0, \tag{21}$$

which is called the *standard power function*.

These functions are self-similar (they have no characteristic scale, which naturally leads to the concept of *fractals*); they have the semigroup property; at the zeros of the gamma function $\Gamma(\lambda + 1)$ they are defined as generalized functions expressed in terms of the δ-function and its derivatives $\delta^{(\lambda)}(t)$; their Laplace transforms also belong to the family of power functions up to a constant factor; in contrast to exponential functions, which have the property of invariance up to a constant factor, power functions do not have this property (hence, the memory property); Tauberian theorems are applicable to them, which allow one to uniquely determine the asymptotic behavior of such functions as $t \to \infty$ from the behavior of the Laplace transform in the region of zero (these theorems are also true under the condition that zero and infinity are interchanged).

Homogeneous functions play a very important role in describing the thermodynamics of phase transitions, in describing the statistical properties of percolation [58,89], in turbulence [56,93], in the modern renormalization group theory of critical phenomena, etc.

Very often, far-reaching conclusions can be drawn from the only premise of the universality of fluctuating systems using *scaling estimates*.

2.3. Probability Power Laws and Non-Gaussian Statistics

Among the objects of the material world, self-similarity, as mentioned above, is very widely represented [6,49,57,58,73,77,89,92,96,100,115]. Power laws are the mathematical expression for self-similarity. These laws obey both objects that grow in size, for example, cities, and objects that break up into separate fragments, for example, stones. The only indispensable condition for the fulfillment of a power-law self-similar law is that *this type of objects does not have an internal scale*. Indeed, there are no real cities with the number of inhabitants less than 1 or more than 10^9. Similarly, the size of a stone cannot be less than a molecule, or more than a continent. Thus, if self-similarity is unlimited, then only in limited areas. The fact that homogeneous power laws do not have natural internal scales leads to another phenomenon – scaling or scale invariance.

We can say that power laws with integer or fractional exponents are *self-similarity generators*. As noted in [96, p. 165]: "Self-similarity, in the end, does not care whether we have an integer exponent or not. Often times, a fractional exponent holds an important clue to solving a convoluted puzzle".

In mathematics, on the basis of power functions, as we have just considered, a fractional calculus is constructed, the concept of poles is introduced and a theory of residues is created, a theory of asymptotic expansions is constructed, and stable distributions are introduced. The penetration of fractional calculus into physics accelerated sharply after the establishment of its close connection with stable distributions of the theory of probability.

The cognitive value of probability theory is revealed only by limit theorems [14]. The interest of classical research was reduced to clarifying the conditions for the convergence of distribution functions of sums of independent random variables to a Gaussian law. Therefore, the classical theory of probability studied only one limiting distribution law – the Gaussian one. In the theory of probability, in parallel with the completion of the classical problematics, the question arose of which laws, in addition to Gaussian, can be limiting for sums of independent random variables. It turned out that the class of limit laws is far from being exhausted by the Gaussian law [14,20,88,91].

The modern theory of probability is based on limit theorems on the convergence of distributions of sums of independent random variables to the so-called stable distributions: Gaussian or non-Gaussian. The former are based on the central limit theorem, and the latter (non-Gaussian) - on the limit theorem proved by B.V. Gnedenko (1939) and V. Doblin (1940) [6,14,20,91].

In this case, the limit theorem imposes restrictions on the form of non-Gaussian distributions. Namely: for the distribution law $F(x)$ to belong to the region of attraction of a stable law with a characteristic exponent α ($0 < \alpha < 2$) different from the Gaussian, it is necessary and sufficient that

$$\frac{F(-x)}{1-F(x)} \to \frac{c_1}{c_2} \text{ as } x \to \infty, \tag{22}$$

for every constant $k > 0$

$$\frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx)} \to k^{\alpha} \text{ as } x \to \infty, \tag{23}$$

where $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$, $0 < \alpha < 2$.

To prove (22) and (23) it is necessary and sufficient that for some selection of constants B_n, the following conditions are satisfied [14, p. 189]:

$$\begin{gathered} nF(B_n x) \to \frac{c_1}{|x|^{\alpha}},\ (x<0), \\ n[1-F(B_n x)] \to \frac{c_2}{x^{\alpha}},\ (x>0), \\ \lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\, n\left\{ \int\limits_{|x|<\varepsilon} x^2 dF(B_n x) - \left[\int\limits_{|x|<\varepsilon} x dF(B_n x) \right]^2 \right\} = 0. \end{gathered} \tag{24}$$

The smaller the value of α, the longer the tail of the distribution and the more it differs from the Gaussian one. For $1 < \alpha < 2$, stable laws have a mathematical expectation; for $0 < \alpha \leq 1$, stable laws have neither variances nor mathematical expectations. Conditions (22)-(24) define the so-called non-Gaussian statistics.

The general theory of stable distributions is comparatively little known to applied specialists due to its complexity and, as was thought earlier, its purely mathematical nature. By definition, a distribution is considered stable if the composition of two such distributions leads to a distribution of the same type. This property can be viewed as a kind of self-similarity. The complexity of their use also lies in the fact that they, as a rule, are not expressed explicitly, but only through characteristic functions.

The mechanism for the formation of non-Gaussian laws does not yet have an unambiguous solution. The most common hypotheses are the following [58,73]: "the principle of least effort" – for Zipf's law, "compromise" structures – for Bradford's law, the interaction of two opposing processes (growth and limitation) and the "thermodynamic" or variational approach - for Zipf-Pareto law. (Note that the first works on the connection of the Zipf-Pareto law in linguistics and economics with stable non-Gaussian distributions belongs to B. Mandelbrot).

Stable laws play the same role in the summation of independent random variables with infinite variances as the usual Gaussian law for finite variances. The system-wide universal character of such non-Gaussian laws was established, first of all, in social and information complex systems and is associated with human behavior. Since complex systems possess structure, the analysis of such systems should take into account both aspects: the randomness of the scatter of variables and the determinism of the structures of the corresponding formations.

2.4. Fractional integroderivatives

Fractional mathematical analysis has a long history and extremely rich content [40,86]. The idea of generalizing the notion of differentiation $d^n f(x)/dx^n$ to noninteger values of n arose from the very inception of differential calculus. At the present time, in fact, there is not a single area of classical analysis that has not been touched on by fractional analysis. The mathematical language of operators of fractional integrodifferentiation is indispensable for describing and studying physical fractal systems, stochastic transfer processes (various relaxation and diffusion processes). Active attempts are being made to explain power-law dependences with fractional exponents (i.e., fractal form) by solutions of equations in fractional derivatives. Works in this direction are, apparently, restrained only by exoticism and the absence of a clear physical interpretation of fractional derivatives and fractional integrals. The apparatus of fractional derivatives and integrals is used in physics, mechanics, chemistry, hydrology, the theory of gravity, etc. (see for example [4,9,10,19,30,31,35,38,40,57,58,79,80,83,86,88,100-103,111,114,117-122,135,136]). This numerous reference was made by the author specifically to show that the applications of this mathematical apparatus are too numerous to list them all.

The time has come to apply the apparatus of fractional derivatives and integrals to the problems of fractal radiophysics and fractal radar [52,57,58,62,82,83]. To do this, first consider some of the fundamental questions of fractional calculus, which are necessary for further use.

Brief historical information. Interest in fractional mathematical analysis arose almost simultaneously with the appearance of classical analysis (even G. Leibniz mentioned this in letters to G. Lopital in 1695 when considering differentials and derivatives of order ½). Probably the earliest more or less systematic study of this issue dates back to the 19th century. and belongs to N. Abel (1823), J. Liouville (1832), B. Riemann (1847) and H. Holmgren

(1864), although earlier contributions were made by L. Euler (1730) and J. Lagrange (1772).

It was in his cycle of works that J. Liouville (1832-35), using the expansion of functions in power series, determined the "q"-th derivative by term-by-term differentiation. He, in particular, gave the first practical applications of the theory he created to the solution of problems in mathematical physics. Then B. Riemann (1847) proposed a different solution based on a definite integral, suitable for power series with noninteger exponents. This work, performed by Riemann in his student years, was published only in 1876 (10 years after his death). The Liouville and Riemann constructions are the main forms of fractional integration. Developing Liouville's idea, A. Grunwald (1867) introduced the concept of a fractional derivative as the limit of difference relations.

In parallel with theoretical beginnings, applications of fractional analysis to the solution of various problems were developed. One of the first such applications was the discovery of N. Abel (1823), who showed that the solution of the tautochron problem can be obtained by an integral transformation, which is written as a derivative of a half-integer order. There is a historical misconception that Abel solved the problem only with an index value of ½. In fact, as noted in [86,136], Abel considered the solution in the general case, and his work played a huge role in the development of the ideas of fractional integrodifferentiation. Holmgren's merit is the consideration of fractional differentiation as an operation inverse to integration and the application of these concepts to the solution of ordinary differential equations.

Special mention should be made of the cycle of works by Corr. Memb. Petersburg Academy of Sciences (1884) A.V. Letnikov (1837-88), who, during his 20-year scientific activity, developed a complete theory of differentiation with an arbitrary pointer (at present, his works are almost completely forgotten) [25]. The works of A.V. Letnikov remained almost unknown abroad. During the period under review in Russia, for the works of A.V. Letnikov's work was followed by N.Ya. Sonin and P.A. Nekrasov. The names of these Russian scientists are also associated with the extension of the Cauchy formula for analytic functions in the complex plane to non-integer values of the integrodifferentiation index.

While recognizing the importance of the works of the above-mentioned scientists, it is necessary, however, to note that fractional calculus became a rigorous mathematical theory only starting with the works of A.V. Letnikov [80].

At the end of the XIX century. A substantial work by J. Hadamard (1892) was published, in which, on the basis of the expansion in a Taylor series, the fractional differentiation of a function analytic in a circle with respect to the radius was considered, which is called the Hadamard approach.

In the first half of the XX century. G. Hardy, G. Weil, M. Riess, P. Montel, A. Marshaud, D. Littlewood, Ya. Tamarkin, E. Post, S.L. Sobolev, A. Sigmund, B. Nagy, A. Erdelyi, H. Kober, J. Cossard, and a number of other scientists. In 1915 G. Hardy and M. Riess used fractional integration to sum up divergent series. In 1917, G. Weil defined fractional integration for periodic functions in the form of a convolution with some special function. An analogue of S.N. Bernstein for fractional derivatives of algebraic polynomials on a finite segment was given in 1918 by P. Montaigne. A. Marshaud (1927) introduced a new form of fractional differentiation, which is applicable in the case of functions with "bad" behavior at infinity. Fractional derivatives of Marshaud were introduced into use. In the works of M. Riess (1936, 1938, 1949), operators of the type of potential (Riess potentials) were obtained, which made it possible to determine the fractional integration of functions of several variables. For some integral operators and integral equations, fractional integrals of Erdelyi and Kober (1940), etc., turned out to be very useful.

Especially for radio physicists and radio engineers, we note the fact that the operational calculus developed by O. Heaviside (1892, 1893, 1920) turned out to be an important stage in the application of generalized derivatives. It was O. Heaviside (1920) who applied fractional differentiation in the theory of transmission lines. After that, other theorists recognized the advantages of this approach and began to develop it in accordance with the accepted mathematical concepts (N. Wiener, J. Carson (1926)).

Abel's equation. The notion of fractional integration is closely related to the Abel integral equation

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}}dt = f(x),\ x > a, \tag{25}$$

where $0 < \alpha < 1$; $\Gamma(\alpha)$ is a gamma function. The solution to equation (25) has the form

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt. \tag{26}$$

For the Abel equation of the form

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt = f(x),\ x < b, \tag{27}$$

there is a treatment formula

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(x-t)^{\alpha}}dt. \tag{28}$$

Using the method of mathematical induction, we prove a formula for an n-fold integral of the form

$$\int_a^x dx \int_a^x dx ... \int_a^x \varphi(x)dx = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt. \tag{29}$$

Noticing that $(n-1)! = \Gamma(n)$, the right-hand side of (29) can be given a meaning even for noninteger values of n.

Fractional Riemann-Liouville and Marshaud operators. Fractional Riemann-Liouville integrals of fractional order ($\alpha > 0$) are

$$(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt,\ x > a, \tag{30}$$

$$\int_a^b \varphi(x)(I_{a+}^{\alpha}\psi)(x) = \int_a^b \psi(x)(I_{b-}^{\alpha}\varphi)(x)dx,\ x < b, \tag{31}$$

The first of them is sometimes called left-sided, and the second – right-sided. Most often they deal with left-hand fractional integration. The operator I_{a+}^{α} in the English-language literature is denoted in the form ${}_aD_x^{\alpha}$ when the sign of α is replaced in (30) by the opposite one, i.e. for $\alpha < 0$.

The fractional integration by parts formula has the form

$$\int_a^b \varphi(x)(I_{a+}^{\alpha}\psi)(x) = \int_a^b \psi(x)(I_{b-}^{\alpha}\varphi)(x)dx. \tag{32}$$

Fractional integration has the semigroup property:

$$I_{a+}^{\alpha}I_{a+}^{\beta}\varphi = I_{a+}^{\alpha+\beta}\varphi,\quad I_{b+}^{\alpha}I_{b+}^{\beta}\varphi = I_{b+}^{\alpha+\beta}\varphi,\ \alpha > 0, \beta > 0. \tag{33}$$

Fractional differentiation is naturally introduced as an operation inverse to fractional integration. Therefore, the fractional derivative is established using fractional integration and further – ordinary differentiation. Therefore, the fractional Riemann-Liouville derivatives of order α for $0 < \alpha < 1$ have the form

$$(D_{a+}^{\alpha}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}, \tag{34}$$

$$(D_{b-}^{\alpha}f)(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)dt}{(t-x)^{\alpha}}. \tag{35}$$

This definition shows that fractional differentiation is non-local. If fractional integrals are defined for any order ($\alpha > 0$), then fractional derivatives are so far only for order ($0 < \alpha < 1$). For large orders ($\alpha \geq 1$) with their integer – $[\alpha]$ and fractional – $\{\alpha\}$ ($0 \leq \{\alpha\} < 1$) parts of the number $\alpha = [\alpha] + \{\alpha\}$ we have

$$D_{a+}^{\alpha}f = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}},\ n = [\alpha]+1, \tag{36}$$

$$D_{b-}^{\alpha} f = -\frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b \frac{f(t)dt}{(t-x)^{\alpha-n+1}},\ n = [\alpha]+1. \quad (37)$$

If α is an integer, then the fractional derivative of order α is understood as the usual differentiation

$$D_{a+}^{\alpha} = \left(\frac{d}{dx}\right)^{\alpha},\ D_{b-}^{\alpha} = \left(-\frac{d}{dx}\right)^{\alpha},\ \alpha = 1,2,3,... \quad (38)$$

Sometimes they also use notation $D_{a+}^{\alpha} f = I_{a+}^{-\alpha} f = (I_{a+}^{\alpha})^{-1} f,\ \alpha > 0,$ meaning by each of them the derivative (34) and (36). Symbols $D_{b-}^{\alpha} f = I_{b-}^{-\alpha} f$ are understood in a similar way.

As an example, consider the power functions $\varphi(x) = (x-a)^{\beta-1}$ and $\varphi(x) = (b-x)^{\beta-1}$, Re$\beta > 0$. For them, the fractional integrals are, respectively,

$$I_{a+}^{\alpha}\varphi = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1};\ I_{b-}^{\alpha}\varphi = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}. \quad (39)$$

Fractional integrals (30) and (31) easily extend from the finite segment $[a, b]$ to the semiaxis (a, ∞) or $(-\infty, b)$. For the general case, when $-\infty < x < \infty$, the fractional integrals along the whole line have the form

$$(I_{+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt;\ (I_{-}^{\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{\varphi(t)}{(t-x)^{1-\alpha}} dt. \quad (40)$$

Similarly to (34) and (35), the Liouville derivatives are introduced

$$(D_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(t)dt}{(x-t)^{\alpha}}, \quad (41)$$

$$(D_{-}^{\alpha} f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} \frac{f(t)dt}{(x-t)^{\alpha}}, \quad (42)$$

where $0 < \alpha < 1$ and $-\infty < x < \infty$. For $\alpha \geq 1$ with $n = [\alpha] + 1$ we have

$$(D_{\pm}^{\alpha} f)(x) = \frac{(\pm 1)}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^{\infty} t^{n-\alpha-1} f(x \mp t) dt. \quad (43)$$

Fractional Liouville derivatives on the axis can be reduced to a more convenient form than (41) and (42). The resulting constructions are called fractional derivatives of Marshaud:

$$(D_{+}^{\alpha} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt, \quad (44)$$

$$(D_{-}^{\alpha} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt, \quad (45)$$

where $0 < \alpha < 1$ and $-\infty < x < \infty$.

In the theory of fractional integrodifferentiation of functions of several variables, which is a fractional power $(-\Delta)^{\alpha/2}$ of the Laplace operator, fractional Riess integrodifferentiation is widely used. In Fourier images $\hat{F}$, this operation is written in the form

$$(-\Delta)^{\alpha/2} f = \hat{F}^{-1} |x|^{-\alpha} \hat{F} f = \begin{cases} I^{\alpha} f, \mathrm{Re}\,\alpha > 0, \\ D^{-\alpha} f, \mathrm{Re}\,\alpha < 0. \end{cases} \quad (46)$$

A detailed exposition of the theory of Riess differentiation is given in [131]. As follows from the convolution theorem for functions, the fractional-order integral $(I_{0+}^{\alpha} f)(x)$, Re $\alpha > 0$ is a Laplace convolution of the form

$$(I_{0+}^{\alpha} f)(x) = \left[f(x) \frac{x_{+}^{\alpha-1}}{\Gamma(\alpha)} \right],\ \mathrm{Re}\ \alpha > 0, \quad (47)$$

as a consequence of which the property of the joint action of the Laplace transform and the operator of fractional integration takes place

$$(LI_{0+}^{\alpha})(p) = p^{-\alpha}(Lf)(p). \quad (48)$$

Generalized Leibniz rule. Let us formulate the generalized Leibniz rule:

$$D_{a+}^{\alpha}(fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D_{a+}^{\alpha-k} f) g^{(k)}, \alpha \in R^1, \quad (49)$$

$$D_{a+}^{\alpha}(fg) = \sum_{k=-\infty}^{\infty} \binom{\alpha}{k+\beta} (D_{a+}^{\alpha-\beta-k} f)(D_{a+}^{\beta+k} g), \quad (50)$$

where

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} = \frac{\sin[(\beta-\alpha)\pi]}{\pi} \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+1)}$$

is the generalized binomial coefficient; $\alpha, \beta \in R^1,\ R^1 = (-\infty, \infty)$; with non-integer β.

Along with the last formulas, we have the Leibniz formula with remainder

$$D_{a+}^{\alpha}(uv) = \sum_{k=0}^{n-1} \binom{\alpha}{k} D_{a+}^{\alpha-k} u v^{(k)} + R_n, \quad (51)$$

$$R_n = \frac{(-1)^n}{\Gamma(-\alpha)(n-1)!} \int_a^x (x-t)^{-\alpha-1} u(t) dt \int_t^x (x-\xi)^{n-1} v^{(n)}(\xi) d\xi, \quad (52)$$

which does not require the function v(x) to be infinitely differentiable.

Results. The operator of integro-differentiation in the sense of the Riemann-Liouville fractional order $\alpha \in$ R with origin at the point α is defined as follows [83]:

$$ {}_{RL}D_{at}^{\alpha} f(t) = \frac{sign(t-a)}{\tilde{A}(-\alpha)} \int_a^t \frac{f(\tau)}{|t-\tau|^{\alpha+1}} d\tau, \ \alpha<0, \quad (53)$$

$$ {}_{RL}D_{at}^{\alpha} = f(t) = f(t), \ \alpha = 0, \quad (54)$$

$$ {}_{RL}D_{at}^{\alpha} = sign^n(t-a) \frac{d^n}{dt^n} D_{at}^{\alpha-n} f(t) = $$
$$ = \frac{1}{\tilde{A}(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (55)$$
$$ n-1 < \alpha \le n, \ n \in N. $$

For functions differentiable on the interval [a, b], the definition of fractional derivatives in the sense of Riemann-Liouville and A.V. Letnikov are equivalent [86,136].

Currently, *Caputo's formulation* is widely used [12,103,104]:

$$ {}_{C}D_{at}^{\alpha} f(t) = sign^n(t-a) \, {}_{RL}D_{at}^{\alpha-n} f^{(n)}(t), $$
$$ n-1 < \alpha \le n, \ n \in N. \quad (56)$$

The Riemann-Liouville and Caputo derivatives are related by the ratio:

$$ {}_{C}D_{at}^{\alpha} f(t) = {}_{RL}D_{at}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\tau)}{\tilde{A}(k-\alpha+1)} |\tau - t|^{k-\alpha}, \quad (57)$$
$$ n-1 < \alpha \le n, \quad n \in N. $$

In the case $\alpha = n$ we get:

$$ {}_{RL}D_{at}^{n} f(t) = {}_{C}D_{at}^{n} f(t) = sign^n(t-a) \frac{d^n}{dt^n} f(t), \ n \in N. \quad (58)$$

The Caputo derivative has the same physical interpretation as the Riemann-Liouville derivative. In particular, for $f(0) = 0$ and $0 < \alpha < 1$, we have the exact equality:

$$ {}_{C}D_{0t}^{\alpha} f(t) = {}_{RL}D_{0t}^{\alpha} f(t). \quad (59)$$

When comparing these derivatives, it is necessary to pay attention to the fact that to calculate the Riemann-Liouville derivative it is necessary to know the values of the function, and for the Caputo derivative – its derivatives, which is much more complicated. Some advantage of the Caputo derivative is that it is equal to zero for a constant function, which is more familiar to a researcher.

Note that in [32] a biography is presented and the surviving works of the Soviet mechanic Alexei Nikiforovich Gerasimov (03.24.1897–03.14.1968) are presented, 20 years earlier than Caputo, who proposed the use of the fractional derivative for viscoelasticity problems (i.e., the Gerasimov-Caputo derivative).

In [112], a new fractional operator is introduced that defines the *local fractional Kolvankar derivative* using the following limit:

$$ D^q f(y) = \lim_{x \to y} \frac{d^q (f(x) - f(y))}{d(x-y)^q}, \quad (60)$$

if this limit exists.

The use of this operator allows the derivative to return the locality property, which is lost when passing from integers to fractional values of the orders of differentiation. Thus, a direct connection is established between the properties of local fractional differentiability and the fractal dimension of nondifferentiable functions, which is illustrated in [112] using the example of the classical nondifferentiable Weierstrass function and Levy flights/motions.

2.5. Fox function

For integral transformations in diffusion processes in fractal space and using fractional operators, the Fox H-function H_{pq}^{mn} is widely used, where $0 \le m \le q$, $0 \le n \le p$ [116]. The importance of the Fox function is that it includes almost all the special functions that go into applied mathematics and statistics as its special cases. Even functions such as the Wright generalization of the Bessel function, the Meyer G-function, or the generalized Maitland hypergeometric function are covered by the Fox class of functions. In addition, a connection has been established between stable laws and Fox functions: the analytical form of a stable law is given through the Fox function [118].

The Fox function is defined as

$$H_{pq}^{mn}\left(z\left|\begin{matrix}(a_1,\alpha_1),...,(a_p,\alpha_p)\\(b_1,\beta_1),...,(b_q,\beta_q)\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_C h(s)z^s ds, \quad (61)$$

$$h(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{i=1}^{n}(1-a_i+\alpha_i s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}, \quad (62)$$

where $\Gamma(s)$ is the gamma function; parameters α_i ($i = 0, 1,\dots, p$) and β_j ($j = 0, 1,\dots, q$) are positive numbers; a_j and b_j are complex numbers satisfying the condition $\alpha_i(b_h + \delta) \neq \beta_h(a_i - 1 - \lambda)$; δ and $\lambda = 0, 1,\dots$; $h = 1, ..., m$; $i = 1,\dots, n$.

The contour of integration C in the complex s-plane passes so that the poles of $\Gamma(b_j - \beta_{jS})$ ($j = 0, 1, ..., m$) are on the right, and the poles $(1 - a_i + a_{iS})$ ($i = 1, ... , n$) – to the left of the contour.

The Laplace transform of the Fox function is also a Fox function, but with different indices:

$$\tilde{H}(p)=\frac{1}{p}H_{q,p+1}^{n+1,m}\left(p\left|\begin{matrix}(1-b_j,\beta_j)\\(1,1),(1-a_i,\alpha_i)\end{matrix}\right.\right),\ 0\le\mu\le 1, \quad (63)$$

$$\tilde{H}(p)=\frac{1}{p}H_{p+1,q}^{m,n+1}\left(\frac{1}{p}\left|\begin{matrix}(0,1),(a_i,\alpha_i)\\(b_j,\beta_j)\end{matrix}\right.\right),\ \mu\ge 1. \quad (64)$$

For the inverse Laplace transform, we have

$$\tilde{H}(t)=\frac{1}{t}H_{q,p+1}^{n,m}\left(t\left|\begin{matrix}(1-b_j,\beta_j)\\(1-a_i,\alpha_i)(1,1)\end{matrix}\right.\right),\ 0\le\mu\le 1, \quad (65)$$

$$\tilde{H}(t)=\frac{1}{t}H_{p+1,q}^{m,n}\left(\frac{1}{t}\left|\begin{matrix}(a_i,\alpha_i)(1,1)\\(b_j,\beta_j)\end{matrix}\right.\right),\ \mu\ge 1. \quad (66)$$

The function $H(z)$ is an analytic function of z under the following conditions: $z \neq 0$ for $\mu > 0$ and $|z| < \beta^{-1}$ for $\mu = 0$, where

$$\mu=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i,\ \beta=\prod_{i=1}^{p}\alpha_i^{\alpha_i}\prod_{j=1}^{q}\beta_j^{\beta_j}. \quad (67)$$

The asymptotic expansion of the Fox functions is given by the expression

$$H_{p,q}^{m,n}(z)-\sum res[h(s)z^s], \quad (68)$$

valid for $\mu > 0$ and $n \neq 0$ for $|z| \to \infty$ on each closed sector $|\arg z| \le \pi\lambda/2$. In this case, the residues must be determined at the points $s = (a_i - 1 - \nu)/\alpha_i$; $i = 1, ..., n$; $\nu = 0, 1, ...$

2.6. Historical overview of undifferentiated functions

In his letter dated January 15, 1898 to F. Klein, L. Boltzmann specially noted that "*in Nature there are such physical problems (statistical mechanics), for the solution of which nondifferentiable functions are absolutely necessary, and if K. Weierstrass had not invented such functions , then physicists simply would have had no choice but to invent them themselves* ". At present, such undifferentiable curves are usually called *fractal* or simply *fractals*. Here is a brief historical overview of such mathematical objects, based on the sources [26,27,58,106,110].

It is also noteworthy that the *concept of self-similarity* entered mathematics from two independent directions (through Cantor sets and Weierstrass functions) at approximately the same time for the basic concepts of mathematics: numbers and functions. Recall that G.V. Leibniz in his treatise "Monadology", written in 1714, used the concept of *self-similarity* ("worlds within worlds"), and also applied it in the definition of the straight line.

After the discovery of differential calculus, it was intuitively formed that each function can be differentiated any number of times. In 1806, Ampere made an attempt to theoretically justify this belief on a purely analytical basis within the framework of the mathematical concepts of Lagrange. Later, some mathematicians automatically transferred Ampere's statements to functions that are continuous in the present sense, while others, considering it the foundation of the whole differential calculus, presented their proofs of this statement and used it to establish other results. Among them are Lacroix (1810), Galois (1831), Raabe (1839), Duhamel (1847), Lamarlet (1855), Freycinet (1860), Bertrand (1864) , Serre and Rubini (1868).

However, the time of the faith of mathematicians about the inextricable connection between the continuity of functions and its differentiability was running out. In 1830, B. Bolzano, in his manuscript

"The Doctrine of Function", constructs the first example of a continuous, nowhere, undifferentiable function. This Bolzano manuscript was discovered only after the First World War around 1920 in the Vienna State Library. Only a hundred years later, his work appeared in print. In 1834-35 the concepts of differentiability and continuity are clearly distinguished by N.I. Lobachevsky. In 1854 Dirichlet notes that in the general case it is impossible to prove the existence of a derivative for an arbitrary continuous function, and expresses his conviction in the existence of a continuous function without a derivative.

In 1861 Riemann gave an example of the function

$$f\left(x\right)=\sum_{n=1}^{\infty}\frac{\sin n^2 x}{n^2}, \qquad (69)$$

which Dubois Reymond claimed to be nondifferentiable on an everywhere dense set.

How difficult the analysis of example (69) turned out to be is evidenced not only by Weierstrass' refusal to carry it out, but also by the fact that before 1916 there was no proof or refutation of the Riemann example. Only in 1916, Hardy, relying on some subtle results of Diophantine analysis, was able to show that (69) has no finite derivative at any point $\xi\pi$, where ξ is an irrational or rational number of the form $2m/(4n + 1)$ or $(2m + 1)/2(2n + 1)$, and m and n are integers; he then generalized somewhat the example of Riemann.

Gerwer expanded this result in 1969 by showing that this function has no finite derivative at the points $\xi\pi$, where ξ is a rational number of the form $(2m + 1)/2^n$, and m, n are integers and $n \geq 1$. He established the existence of a derivative, equal to $-1/2$ at the points $\xi\pi$ when ξ is a rational number with an odd denominator and numerator, so that the Riemann function is differentiable on an infinite set of points. In the next paper, Gerver showed that the Riemann function has no other points of differentiability besides those indicated above.

Until 1870, apart from the Riemann function mentioned above, not a single example of a continuous function having no derivative at an infinite set of points was published. According to Guel, who reviewed Hankel's memoir on such functions, "*today there is not a single mathematician who would believe in the existence of continuous functions without derivatives.*" In 1870, Hankel proposed a method for *condensing singularities,* which consists in constructing a function using an absolutely convergent series, each term of which has a singular point. This is how he obtained examples of continuous functions that have no derivative on an everywhere dense set of rational points. One such example is a function of the form

$$f\left(x\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}\sin\left(\pi n x\right)\sin\left(\frac{1}{\sin\left(\pi n x\right)}\right), \qquad (70)$$

where n is a natural number, $s > 1$.

In 1873, Schwartz constructed another example of a monotone continuous function with no derivative on an everywhere dense set of points:

$$f(x)=\sum_{n=1}^{\infty}\frac{\varphi(2^n x)}{4^n}, \qquad (71)$$

where $\varphi\left(x\right)=\left[x\right]-\sqrt{x-\left[x\right]}$, $x>0$, $[x]$ is the integer part of x.

Schwarz considered this function to be nondifferentiable, but, as it turned out later, it has a finite derivative almost everywhere.

Later, Weierstrass built, as is commonly believed in 1861, his famous function

$$f(x)\equiv W(x)=\sum_{n=1}^{\infty}a^n\cos(b^n\pi x). \qquad (72)$$

Here $0 < a < 1$, $b > 1$ is an odd integer, $ab > 1 + 3\pi/2$. Weierstrass reported to the Berlin Academy of Sciences on July 18, 1872, and the example itself was published only in 1875 by Dubois-Reymond. Therefore, as noted in [115], "*the year 1875 is nothing more than a convenient symbolic date to indicate the beginning of the Great Crisis of Mathematics*."

In the preface to his book, S. Sachs wrote: "*Studies dealing with non-analytical functions and functions that violate those laws that were assumed to be universal, these studies were viewed almost as the spread of chaos and anarchy where previous generations sought order and harmony*." [85]. S. Hermite wrote to T. Stiltjes in 1893: "*With horror and disgust I turn away from this growing ulcer of functions that have no derivative*." Even at the beginning of the XX century. G. Bussensk was not alone in the opinion that "*the whole interest of a function lies in the possession of its derivative,*" meaning the ordinary derivative.

Independently of Weierstrass, Darboux came to the same idea, who generalized the examples of Hankel and Schwarz and constructed the function

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin\left[(n+1)!x\right]}{n!}, \tag{73}$$

which has no derivative for any x. Darboux reported his results at a meeting of the French Mathematical Society on March 19, 1873 and January 28, 1874, i.e. before the publication of Dubois-Reymond's publication. Some details of the priority pick between Weierstrass and Darboux were published in 1973 (*Dugac P.* Elements d 'analyze de Karl Weierstrass // Archive for Hist. Exact. Sci. 1973. V. 10. P. 41-176).

2.7. Dini's theorem on functions without derivatives and the continuation of the survey

The above studies served as the basis for constructing classes of nondifferentiable functions and searching for general conditions for the differentiability of continuous functions. The greatest contribution to this direction was made by the Italian mathematician W. Dini, who came close to Lebesgue's theorem on the derivative of a continuous monotone function. It was he who formulated in 1877, and in 1878 proved a sufficiently general theorem on the existence of continuous functions without derivatives (the statement of Dini's theorem is given by us according to [26]).

Theorem 1. (Dini, 1877). *Let $f_n(x)$ functions be given on $0 \le x \le 1$ satisfying the following conditions:*

1) all functions $f_n(x)$ are continuous and have bounded derivatives;

2) the series $\sum_{n=1}^{\infty} f_n(x)$ converges on [0, 1] to a continuous function f(x);

3) each of $f_n(x)$ has a finite number of extrema, and their number increases indefinitely with n and, moreover, in such a way that for any $\varepsilon > 0$ one can find n_0 such that for $n > n_0$ the distances between the extremum points will be less than ε;

4) if δ_n is the largest distance between two successive extrema, and D_n is the largest difference in absolute value between two successive extreme values, then $\lim_{n\to\infty}(\delta_n / \mathrm{D}_n) = 0$;

5) if we denote by h_n for each x those two increments (one of which is positive and the other negative) for which $x + h_n$ gives the first right (respectively, left) extremum, for which $|f_n(x+h_n)-f_n(x)|\ge\frac{1}{2}D_n$, it is possible to specify such positive numbers r_n that for all $x \in$ [0,1] and such h_n corresponding to each x we have $\left|R_n(x+h_n)-R_n(x)\right|\le 2r_n$, where $R_n(x)$ is the remainder of the series $\sum_{n=1}^{\infty} f_n(x)$ from item 2;

6) if c_n is a sequence of positive numbers such $\left|u_n'(x)\right|<c_n$ that for all $x\in[0,1]$, then starting from some index $\frac{4\delta_n}{D_n}\sum_{\nu=1}^{n}c_\nu+\frac{4r_n}{D_n}\le\theta,\quad 0\le\theta<1;$

7) the sign of the difference $f_n(x+h_n)-f_n(x)$, starting from some n_0, does not depend on h_n for all x and $n > n_0$.

Then the function f(x), defined by the series $\sum_{n=1}^{\infty} f_n(x)$ from item 2, will not have a finite derivative at any point. It can have an infinite derivative on an infinite set of points $x\in[0,1]$.

Then Dini showed that under some additional assumptions such a function *f*(*x*) will not have an infinite derivative at any point. It can be noted that the class of functions satisfying Dini's theorem is infinite; in particular, it contains the Weierstrass function.

In 1879, Darboux proposed a fairly general method for constructing nondifferentiable functions. He studied the functions φ(x) defined by the series

$$\varphi(x)=\sum_{n=1}^{\infty}\frac{f(a_n b_n x)}{a_n}, \quad (74)$$

where a_n and b_n are some sequences of real numbers, $f(x)$ is a continuous bounded function with a bounded second derivative.

If the sequences $\{a_n\}$ and $\{b_n\}$ are chosen so that for fixing k we have

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=0,\ \lim_{n\to\infty}\frac{a_1b_1^2+a_2b_2^2+\ldots+a_{n-k}b_{n-k}^2}{a_n}=0, \quad (75)$$

then series (74) converges everywhere to some continuous function φ(x). With further restrictions on the choice of $\{a_n\}$, $\{b_n\}$, k, and $f(x)$, one can obtain continuous functions that have no derivative at any point. So, if $b_n = 1$, $k = 1$, then on the numbers a_n it is sufficient to impose the condition $\lim_{n\to\infty}\frac{a_1+a_2+\ldots+a_{n-1}}{a_n}=0,$ which is satisfied, for example, the numbers $a_n = n!$, so that one can indicate an infinite set of functions $f(x)$ for which (74) would be just such a function.

Let $f(x) = \cos x$, then φ(x) is nowhere differentiable. Choosing $b_n = n + 1$, $k = 3$ and $f(x) = \sin x$, we obtain the function φ(x) from Darboux's previous work. In 1918 a method for constructing continuous nondifferentiable functions was indicated by *K*. Knopp. We can say that after the above-mentioned works, a whole industry was created for the production of both individual functions and their entire classes.

Note that the example of the Weierstrass function is based on the properties of the lacunary series, i.e. such a series in which nonzero terms are "very sparse and scattered." The concept of a lacunary trigonometric series was introduced by J. Hadamard in 1892 in the study of functions that cannot be analytically extended beyond the boundary of the circle of convergence. A lacunar (in the sense of Hadamard) *trigonometric series* is a series of the form

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos n_k x+b_k\sin n_k x, \quad (76)$$

for $\frac{n_{k+1}}{n_k}\geq q>1.$. Thus, the numbers n_k of the lacunary series (76) for all k grow no slower than a geometric progression with a denominator greater than one. In 1916, J. Hardy also proved that the Weierstrass function $W(x)$ has no finite derivative at any point, provided $a < 1$, $b > 1$ and $ab \geq 1$.

2.8. Many nondifferentiable functions

Let us briefly consider the question of the place occupied by differentiable functions in the set of all continuous functions. A set X of a topological space M is *a set of the first category* on M if it is the union of a countable family of sets that are nowhere dense on M. *Sets of the second category* are defined as sets that are not sets of the first category. These definitions were formulated in 1899 by Baer [11].

By Baire's theorem, the complement of any set of the first category on the line is dense. No interval on the set of real numbers ℝ is a set of the first category. Each countable set will be a set of the first category and a set of measure zero. In the set of real numbers, rational numbers form a set of the first category. The simplest example of an uncountable set belonging to a set of the first category and a set of measure zero is the Cantor set, which has the cardinality of the continuum. It can be proved that the line can be split into two complementary sets A and B so that A is a set of the first category, and B has measure zero. In many problems of topology and theory of functions, *sets of the first category play a role analogous to the role of sets of measure zero in measure theory* (sets that can be "neglected").

At present, sets of the second category are defined according to Baire, and the complement to a set of the first category is called *a residual set*. When proving existence theorems in set theory, the *method of categories* is often used, which is based on Baire's theorem, according to which every complete metric space is a set of the second category in itself. Based on this, it is proved

Theorem 2 (Banach, S. Mazurkevich; 1931). *Let C be the space of continuous functions x with period*

1, endowed with the norm $\|x\| = \max|x(t)|,\ 0 \le t \le 1$. *Let T be the set of functions from C that do not have a finite right derivative at any point* $t \in [0,1]$. *Then T is a set of the second Baire category on C, and its complement is a set of the first category.*

Consequently, the set of functions that have a finite one-sided derivative at least at one point $t \in [0,1]$ is negligible in the sense of the Baire category in comparison with the set of all continuous functions. This is all the more true for functions with a finite ordinary derivative.

Classes of continuous functions without derivatives considered in the 19th century. and in the first two decades of the XX century, did not give an example of such a singular continuous function, in which a finite or infinite one-sided (left or right) derivative did not exist at any point (the Weierstrass function (72), for example, has a one-sided derivative everywhere dense set). The first example of such a strongly nondifferentiable function was constructed in 1922 (published in 1924) by A.S. Besicovich.

In this regard, Banach and Steinhaus raised the question of extending, using the method of categories, the result of S. Mazurkiewicz and Banach to functions of Besicovitch type: is it possible to show that the complement of the set of all continuous functions that have neither finite nor infinite derivative at any point , is the set of the first category?

In 1932 Sachs gave a negative answer to this question. He showed that the set of continuous functions on [0, 1], which either have a finite right derivative, or this derivative is equal on the set of cardinality to the continuum, is a set of the second category in the space of all continuous functions. Thus, *the class of functions that are one-sided differentiable at least at one point, in the sense of categories, is substantially broader than the class of functions that have an ordinary derivative at least at one point.*

Accordingly, the class of functions that have neither a finite nor an infinite one-sided derivative at every point of the domain is narrower in the same sense of the class of functions that nowhere have a two-sided derivative. According to Sachs, "*This may explain the difficulty in finding the first example of a function that does not have a finite or infinite one-way derivative at every point.*" At the same time, Sachs' result indicated an essential difference between the operators of one-sided and two-sided differentiation.

In order to expand the known classes of nondifferentiable functions, V. Orlicz in 1947 found sufficiently general conditions *under which continuous functions that are sums of uniformly converging series are nowhere differentiable.* However, the generality of the results obtained was achieved due to the fact that the coefficients of these series were set ineffectively, using the method of categories. Orlich himself described this approach as "in a sense intermediate" between "effective" methods of specifying nondifferentiable functions in the form of series and the "ineffective" method of S. Mazurkevich-Banach.

2.9. Stationarity and nondifferentiable functions

Thus, *the class of continuous functions that have no derivative at any point is immeasurably richer than the class of functions with derivatives.* As aptly noted in [27, p. 222], "*A curious situation arose when it turned out that those continuous functions that have been studied by mathematicians for centuries, those that have been used to describe the phenomena of the external world, - these functions belong only to a negligible class of all continuous functions.*" Gradually, mathematicians got used to the fact that nowhere differentiable functions really exist, but physicists did not agree with this for a long time and perceived such functions as ugly products of mathematical fantasy that have no relation to the real world (they proceeded from the principle "in physics all functions are differentiable").

From the standpoint of modern science, *a function without a derivative is not at all an abstract concept, but the trajectory of a Brownian particle.* As noted in the 20s. XX century. N. Wiener: "*Within the framework of this theory, I was able to confirm*

Perrin's remark by showing that, with the exception of many cases with a total probability of zero, all trajectories of Brownian motion are continuous curves that are not differentiable anywhere."

It is essential that *in the spectral theory of stationary random processes*, nondifferentiable functions arise in a completely natural way and can be avoided only if the stationarity condition, which has a clear physical meaning, is abandoned, which only makes this theory simple and clear [98]. Let us briefly explain this fact.

In the spectral expansion of a stationary process $X(t)$, the use of the Stieltjes integral turns out to be inevitable, since the random function $Z(\omega)$ is not differentiable in any sense and therefore it is impossible to pass from the Fourier-Stieltjes integral

$$X(t)=\int_{-\infty}^{\infty} e^{i\omega t}\,dZ(\omega) \tag{77}$$

to the usual Fourier integral. In the case of the existence of the spectral density $f(\omega)$ always

$$\left\langle \left|dZ(\omega)\right|^2 \right\rangle = f(\omega)d\omega. \tag{78}$$

By virtue of (78), in all real physical cases, when the process $X(t)$ corresponds to a positive spectral density $f(\omega)$, the mean square of the increment $\Delta Z(\omega)$ of the function $Z(\omega)$ on a small segment $\Delta\omega$ of the frequency axis will be close to $f(\omega)\Delta\omega$, that is, has the same order of smallness as $\Delta\omega$. In this case, the value of $\Delta Z(\omega)$ itself is, as a rule, of order $(\Delta\omega)^{1/2}$, which is incompatible with the assumption that the function $\Delta Z(\omega)$ is differentiable, i.e. on the existence of a limit of the ratio $\Delta Z(\omega)/\Delta\omega$ as $\Delta\omega \to 0$.

As noted in [98, p. 113], "*we are faced here with a rather rare case when in a problem that has a real physical meaning, nowhere differentiable functions appear, which until quite recently were considered by many applied scientists to be an abstruse mathematical abstraction that cannot have any applications*".

In the arsenal of mathematics, there was also an analytical apparatus for describing such objects and processes. The place of the usual dimension was taken by *the Hausdorff dimension*, and the place of derivatives was taken by *the fractional derivative* or the *Hölder exponent*.

2.10. Examples of constructing some nondifferentiable functions

Here are some examples of constructing nondifferentiable functions [57,58,106,110,115].

Graphs of Riemann, Weierstrass and Takagi functions. Returning to the historical examples of the discovery of functions without derivatives, we note that specific examples of such functions sometimes lead to interesting conclusions. In 1903, the Japanese mathematician Takagi discovered a simple example of a nowhere differentiable function

$$T(x)=\sum_{n\geq 1} 2^{-n}\varphi(2^{n-1}x). \tag{79}$$

Here $\varphi(x)=2\left|x-\left[x+(1/2)\right]\right|$, where $[x]$ is the operation of selecting the integer part of x. The function $T(x)$ is a typical example of "condensation of singularities", since it is a superposition of the so-called *sawtooth functions*. The functions of Riemann (69), Weierstrass (72), and Takagi (79) have peaks at a countable number of points (**Fig. 1**). It should be said that the graphs of such undifferentiated functions are described by an infinite number of infinitely small convolutions ("ripple waves"), but it is almost impossible to give a visual representation of them without distorting their essential features.

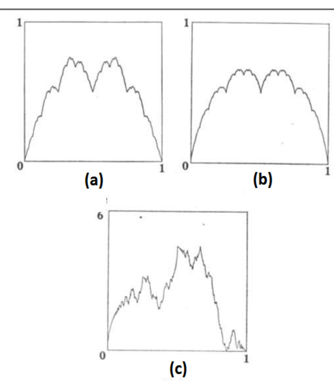


Fig. 1. *Graphs of undifferentiated functions: a - Weierstrass function (1 – W(x))/2 – for values b = a⁻¹ = 2; b –Takagi function; c – Riemann function.*

E. Hobson also studied a number of

$$\sum_{n\geq 1} a^n \varphi(b^n x),\ 0<a<1, \tag{80}$$

and showed on the basis of Knopp's method that the conditions $ab > 4$ (when b is an even integer), or $ab > 1$ (when b is an odd integer) forbid the existence of a finite or infinite derivative. For $b = a^{-1} = 10$, this is an example given by Van der Waerden in 1930 (see below). De Ram also pointed out that if we take $b = a^{-1}$, considering b to be an even integer, then the series (80) will not have a finite derivative.

Construction of the Bolzano function. Now consider the construction of the Bolzano function $B(x)$. Let us define an auxiliary function $B_n(x)$. Consider the construction shown in **Fig. 2***a* and the fact that the segment AB is replaced by the broken line $ACDEB$ with the following coordinates of the points: $A[p, q]$, $C[p + (\delta/4, q - (\Delta/2)]$, $D[p + (2\delta/4), q)]$, $E[p + (3\delta/4), q + (\Delta/2)]$, $B[p + \delta, q + \Delta]$. Let the graph of the function $B_0(x)$ be the segment $A_{11}(0,0)$ and $A_{25}(1,1)$; let $A_{11}P = a$, $A_{25}P = h$ (Fig. 2*b*).

Replace $A_{11}A_{25}$ with a broken line $A_{11}A_{22}A_{23}A_{24}A_{25}$ according to the above rule. The coordinates of the characteristic points are: $A_{21} \equiv A_{11}(0,0)$, $A_{22}(1/4 - 1/2)$, $A_{23}(1/2.0)$, $A_{24}(3/4.1/2)$, $A_{25}(1.1)$, which determines the function $B_1(x)$ and its graph $A_{11}A_{22}A_{23}A_{24}A_{25}$. In Fig. 2*b* shows the graphs of the functions $B_0(x)$ and $B_1(x)$, respectively. By the function $B_1(x)$, as shown in Fig. 2*c*, we construct the function $B_2(x)$. In Fig. 2*d* shows the graphs of the functions

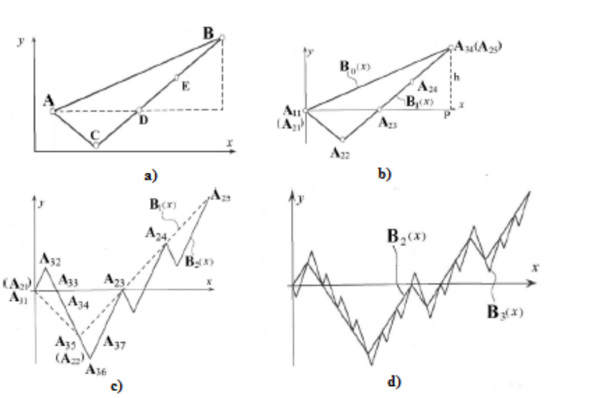


Fig. 2. *Construction of a non-differentiable Bolzano function: a) - the first construction, b) - functions $B_0(x)$ and $B_1(x)$, c) - functions $B_1(x)$ and $B_1(x)$, d) - functions $B_2(x)$ and $B_3(x)$.*

$B_2(x)$ and $B_3(x)$, respectively. Repeating this operation n times, we arrive at the function $B_n(x)$. Oscillation of the function $B_n(x)$ in each of the intervals

$$\left(\frac{s}{4^n}a,\ \frac{s+1}{4^n}a\right),\quad s=0,1,2,\ldots,4^n-1,\ n=0,1,2,3,\ldots \tag{81}$$

will be $\omega_n\left(\frac{s}{4^n}a,\ \frac{s+1}{4^n}a\right)=\frac{h}{2^n}$. In the interval (0, a) for the oscillation $B_n(x)$, we can obtain $\omega_n(0, a) = h(2 - 2^{-n})$.

We now define the Bolzano function $B(x)$ at the points $x = ka/4^n$ for the coefficients $0 \leq k \leq 4^n$, k is an integer, $n = 0, 1, 2, 3, \ldots$, setting $B(x) = B_n(x)$. Then the oscillation $B(x)$ on the set of all considered points $x = ka/4^n$ belonging to one of the intervals (81) will be $\omega\left(\frac{s}{4^n}a,\ \frac{s+1}{4^n}a\right)=h/2^{n-1}$. For values of x different from $t = ka/4^n$, the Bolzano function is determined by the passage to the limit $B(x)=\lim_{t\to x}B(t)$. Oscillation in any interval of length $a/4^n$ satisfies the inequality $\omega\left(x, x+\frac{a}{4^n}\right)>h/2^n$. Thus, the Bolzano function $B(x)$ is defined on the entire interval (0, a) and is continuous on it.

Consider two more other algorithms for synthesizing the Bolzano function. Let each value

$$\frac{x}{a}=\frac{c_1}{4}+\frac{c_2}{4^2}+\ldots+\frac{c_k}{4^k}+\ldots \tag{82}$$

on the interval (0, a) corresponds to

$$\frac{B(x)}{h}=\frac{d_1}{2}+\frac{d_2}{2^2}+\ldots+\frac{d_k}{2^k}+\ldots \tag{83}$$

The numbers d_k are determined by the numbers c_k, according to the rule

$$\frac{c_k \mid 0 \mid}{d_k \mid 0 \mid}\frac{1\mid}{\mp 1\mid}\frac{2\mid}{0\mid}\frac{3\mid}{\pm 1\mid}. \tag{84}$$

Here it is necessary to take the lower signs if among the numbers $c_{k-1}, c_{k-2}, \ldots, c_1$ there is an odd number of them equal to zero, for example:

$$\begin{aligned} &B\left[\left(1/4+2/4^2+1/4^4\right)a\right]=\left(-1/2+1/2^4\right)h,\\ &B\left[\left(1/4^2+1/4^4\right)a\right]=\left(1/2^2+1/2^4\right)h. \end{aligned} \tag{85}$$

Relations (82)-(85) define the Bolzano function.

Consider the third algorithm for constructing the Bolzano function based on the properties of a certain series. Let us define on the segment A_1C of length a with the coordinates of the ends

0 (point A_1) and a (point C) the functions $b_0(x)$, $b_1(x)$, $b_2(x)$ by segment A_1A_5 and broken lines shown in **Fig. 3**.

Function $b_1(x)$ corresponds to a polyline with links formed by the lateral sides of a triangle with base a and height $3b/4$, function $b_2(x)$ is a polyline formed by the lateral sides of four triangles with base $a/4$ and height $3b/8$. Continuing this process, we arrive at the function

$$B_n(x) = \sum_{i=1}^{n} b_i(x). \tag{86}$$

The resulting series converges uniformly as $n \to \infty$, and its sum, equal to $B(x)$, gives us a function that is continuous on $(0, a)$ and has no derivative anywhere in this interval. The extrema of the Bolzano function $B(x)$ are observed at points with abscissas $a(s + 0.25)/4^{n-1}$ for $s = 0, 1, 2, ..., 4^{n-1} - 1$, $n = 1, 2, 3,. ..$ which (abscissas) form an everywhere dense set on the interval (0, a).

Construction of the Besicovitch function. Here are the stages of constructing *the Besicovitch function*. To do this, you need to build a stepped triangle. In **Fig. 4** shows the segment $AB = 2a$ and points $C(a, b)$ and $D(a, 0)$. On the segment AD we construct the segment $l_1 = a/4 = a/2^2$, placing it centrally. Then the segment AD is divided by the segment l_1 into two equal segments; on each of them we place centrally the segments $l_2 = l_3 = a/2^4$.

Segments l_1, l_2, l_3 divide segment AD into four equal segments; on each of them we place the central segments (counting from left to right) $l_4 = l_5 = l_6 = l_7 = a/2^6$, etc. Thus, on the segment AD, an infinite set of segments

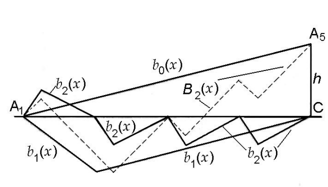


Fig. 3. *The third algorithm for constructing the Bolzano function.*

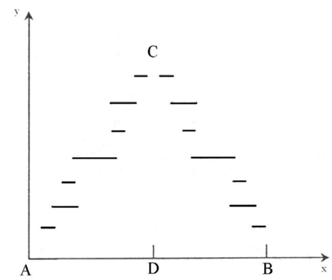


Fig. 4. *Construction of a non-differentiable Besicovitch function.*

l_1, l_2, l_3, ... will be constructed, the union of which L is an everywhere dense set with total length $a/2$. We construct a similar system of segments on the segment DB. Together we will call those and other segments the first series of segments.

We denote by $m(x)$ the Lebesgue measure of the set of points of the interval $(0, x)$ that do not belong to L, that is, measure of the set $\overline{L} \cap (0, x)$. On the segment AD, we define the function $\varphi(x)$ by setting

$$\varphi(x) = \frac{2b}{a} m(x). \tag{87}$$

It follows from (87) that the function $\varphi(x)$ has a constant value on any interval l_i. Thus, points A and C will be connected by some stepped curve; we connect points C and B with the same stepped curve. The resulting figure is called a stepped triangle with base $2a$ and height b.

On all segments of the first series, as on the bases, we construct stepped triangles with their vertices downward – equal on equal bases, choosing the heights so that the apex of the lowest of the equal triangles is on the side AB. The construction of all these triangles was named by A.S. Besicovitch operation of notching the triangle ABC inside. Having performed the same internal notching operation over the resulting infinite series of triangles (the first series), we obtain the second series of triangles. We also subject them to internal serration, etc.

Now we define the function $f(x)$ on the segment AB: **1**) at the points of the segment AB that do not belong to the first series of segments, – the ordinates of the sides of the stepped triangle ABC; **2**) at the points of the segments of the first series that do not belong to the segments of the second series –by the ordinates of the sides of the triangles of the first series; **3**) at the points of the segments of the second series that do not belong to the segments of the third series – by the ordinates of the sides of the triangles of the second series, etc .; **4**) at the points belonging to the segments of all series (they constitute an ensemble of measure zero), according to the principle of continuity.

The Besicovitch function defined in this way is a singular continuous function that has neither right nor left derivative at any point.

Construction of the Van der Waerden function. Consider now the *van der Waerden function.* The idea behind this example is based on the fact that a sequence of integers converges only when all its members, starting with some one, coincide. Let $f_0(x)$ be a function equal to the distance from point x to the nearest integer point:

$$f_0(x) = \begin{cases} x, & 0 \le x \le 1/2, \\ 1-x, & 1/2 \le x \le 1, \end{cases} \quad (88)$$

where $f_0(x + 1) = f_0(x)$ for any real x.

The function $f_0(x)$ is continuous on the entire numerical axis, periodic with a period of 1, linear on each segment $\left[\frac{s-1}{2};\frac{s}{2}\right]$, where s is an integer, and the slope of the graph $f_0(x)$ on each such segment $\pm$ 1. Next, we introduce a sequence of functions f_n (n = 0, 1, 2, ...):

$$f_n(x) = \frac{f_0(4^n x)}{4^n}. \quad (89)$$

For any natural n, the function $f_n(x)$ is continuous, periodic with a period of 4^{-n}, a maximum value of $4^{-n}/2$, linear on each segment $\left[\frac{s-1}{2\cdot 4^n},\frac{s}{2\cdot 4^n}\right]$, and the slope of its graph on each such segment is $\pm$ 1. Finally, we introduce the function

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{f_0(4^n x)}{4^n}. \quad (90)$$

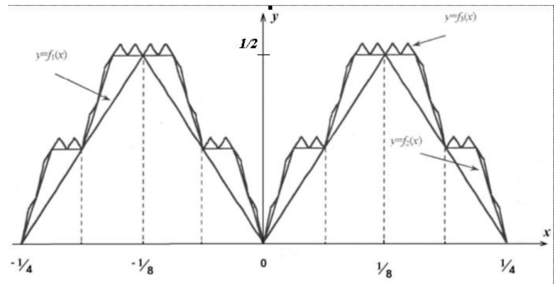


Fig. 5. *The first three partial sums in the case of constructing a non-differentiable Van der Waerden function.*

Since $0 \le f_0(x) \le 4^{-n}$, then by the Weierstrass criterion the series defining $f(x)$ converges uniformly, and the continuity of all $f_n(x)$ implies the continuity of the function $f(x)$. In **Fig. 5** shows the graphs of the functions $f_1(x)$, $f_1(x) + f_2(x)$, $f_1(x) + f_2(x) + f_3(x)$. As the number of terms increases, the number of vertices at which the function $f(x)$ has no derivative increases indefinitely.

2.11. Nondifferentiable functions and functional equations

Let us briefly consider following [58,77,110] the following functional equation:

$$f(x) - af(bx) = g(x). \quad (91)$$

It was de Ram who noticed in 1957 that the Weierstrass function (72) and series (80) satisfy (91) in the case $g(x) = a\cos(b\pi x)$ and $g(x) = a\varphi(bx)$, respectively. The function $\varphi(x)$ is defined above in expression (80). If we put $g(x) = a\cos(b\pi x)$, then equation (91) will have a solution on the interval (–∞, + ∞), depending on an arbitrary function, and the only continuous solution is the Weierstrass function.

G. Faber, considering the function

$$\sum_{n\ge 1} 10^{-n}\varphi(2^{n!}x), \quad (92)$$

showed in 1907 that (92) does not correspond to the Lipschitz condition of any order. Then F. Cater in 1983 investigated the function

$$\sum_{n\ge 1} 2^{-n!}\cos(2^{(2n)!}x) \quad (93)$$

and proved that it has no cusps and has interesting extremal properties.

Consider the following functional equation:

$$\frac{1}{p}\left\{f(\frac{x}{p})+f(\frac{x+1}{p})+...+f(\frac{x+p-1}{p})\right\}=\lambda f(\mu x). \quad (94)$$

This equation was investigated by E. Artin in 1964, characterizing its unique smooth solution by the Euler gamma function. The Japanese mathematician Hata in 1985, based on (94), solved the problem of finding the eigenvalue λ for a certain Parron-Frobenius operator and investigated various solutions (94) with respect to the eigenvalue. He also noted that if $b > 2$ is an integer, then the Weierstrass function $W(2x) + \cos(2\pi x)$ satisfies (94) for $p = b$, $\mu = 1$ and $\lambda = a$; the Takagi function $T(x) - 1/2$ also satisfies (94) for $p = 2$, $\mu = 1$, and $\lambda = 1/2$; and the Riemann function satisfies (94) for $p = 2$, $\mu = 2$, and $\lambda = 1/4$.

Using the concept of a lacunary series (see above), in 1984 (Kaplan et al.) Studied the series

$$f(x)=\sum_{n\ge1}a^n r(b^n x),\ 0<a<1, \quad (95)$$

where $ab > 1$ and $r(x)$ is a quasiperiodic function.

Under certain restrictions on $r(x)$, series (95) is either continuously differentiable or, moreover, not differentiable anywhere. In the latter case, the dimension of the graph of function (95) satisfies the equality

$$D=2+\left[(\log a)/(\log b)\right]. \quad (96)$$

The graph of the function g will have a fractal dimension D greater than one when g is singular. In 1937 A.S. Besicovich showed that if $g(x)$ belongs to the class Lip(δ), $0 <\delta <1$, then the function has a finite k-dimensional measure $k = 2 - \delta$; a function g was also constructed for which the k-dimensional measure is indeed positive for $1 \le k \le 2 - \delta$. At the same time, it was shown more generally that if $x(t)$ belongs to the class Lip(δ) and $y(t)$ belongs to the class Lip(δ'), where $\delta + \delta' > 1$, $0 < \delta' \le \delta \le 1$, then the curve $(x(t), y(t))$ has a finite k-dimensional measure $k = 2 - (\delta + \delta' - 1)/\delta$. In 1945, Klein constructed a curve $(x(t), y(t))$ for which the dimension $k = 2 - (\delta + \delta' - 1)/\delta$ is really attained.

If

$$g(x)=\sum_{n\ge1}\lambda_n^{-s}\varphi(\lambda_n x), \quad (97)$$

where $0 < s < 1$ and $\{\lambda_n\}$ is a sequence of positive numbers satisfying the conditions $\lambda_{n+1}/\lambda_n \to \infty$ and $\log(\lambda_{n+1})/\log(\lambda_n) \to 1$ as $n \to \infty$, then $D = 2 - s$.

However, it is difficult to accurately determine the value of D for the Weierstrass function (72) and sequence (80). It is believed that in both cases

$$D=2+\frac{\log a}{\log b}. \quad (98)$$

This value seems to be quite reasonable, since Hardy showed in 1916 that if $\xi = -(\log a)/(\log b) < 1$, then $W(x + h) - W(x) = O(|x|^\xi)$ and $W(x + h) - W(x) \ne o(|x|^\xi)$, for any value of x.

2.12. Nondifferentiable functions and chaotic mappings

Based on [58,77,110], we present some information about chaotic mappings. Consider a one-dimensional dynamical system described by a one-dimensional logistic map (Verhulst map):

$$y(x)=4x(1-x) \quad (99)$$

on the unit interval **I**. It is well known that the n-fold iteration y^n can be expressed as

$$y^n(x)=\sin^2\left(2^n \arcsin\sqrt{x}\right). \quad (100)$$

Here $y^n(x)$ means the nth iteration of the function $y(x)$, not the nth power of $y(x)$. In 1983, Japanese mathematicians (Yamaguchi and Hata) first proposed to combine y^n with the Weierstrass function (72). In this case, the final dependence is obtained

$$F(a,x)=\sum_{n\ge1}a^n y^n(x)= =\frac{1}{2(1-a)}-\frac{1}{2}\sum_{n\ge0}a^n\cos\left(2^{n+1}\arcsin\sqrt{x}\right), \quad (101)$$

and the generating function $F(a,x)$ is nowhere differentiable for $1/2 \le a <1$.

Similarly, we find

$$F(a,x)=\sum_{n\ge0}a^{\delta}\varphi^n(x)=\sum_{n\ge0}a^n\varphi^n(2^{\ -1}x) \text{ for } x\in \mathbf{I}. \quad (102)$$

When considering (101) and (102), the question arises as to what types of functions ω: **I** → **I** cause the nondifferentiability of their generating function

$$F(a,x)=\sum_{n\geq 0} a^n \omega^n(x) \tag{103}$$

given x. The answer to this question is given by the theorems given in [110].

Weierstrass function (72) for $b = 2$ can also be represented as

$$\sum_{n\geq 0} a^n \cos(2^n \pi x)=\sum_{n\geq 0} a^n \cos(\pi\varphi^n(x)). \tag{104}$$

Consequently, the Weierstrass function and series (80) for $b = 2$ are special cases of the series

$$F(a,x)=\sum_{n\geq 0} a^n g(\varphi^n(x)), \tag{105}$$

where $F(0, x) = g(x)$ is a smooth function on the unit interval **I**.

Series (105) is the only continuous solution to the functional equation

$$F(a,x)-aF(a,\varphi(x))=g(x). \tag{106}$$

For series of type (105), *a replacement operator* S_ω was introduced in the form

$$S_\omega(f)(x)=f(\omega(x)), \tag{107}$$

for $x \in$ **I**. In this case, series (105) is written as

$$F(a,x)=\sum_{n\geq 0} a^n S_\varphi^n(g)=(Id-aS_\varphi)^{-1}(g), \tag{108}$$

where the operator $(Id - aS_\varphi)^{-1}$ is known as the resolvent of the operator S_φ.

Therefore, the operator $(Id - aS_\varphi)^{-1}$ maps $g_0(x) = \cos\pi x$ to the Weierstrass function, and $g_1(x) = x$ to series (80) for $b = 2$, i.e. it maps some smooth function onto a function that is not differentiable anywhere. Further mathematical formalism of the replacement operator is beyond the scope of our description and is presented in detail in [110].

Takagi function (79) and series (102) are special cases of a function of the form

$$f(x)=\sum_{n\geq 0} c^n \varphi^n(x). \tag{109}$$

Although there are no simple functional equations that the series (109) as a whole must satisfy, it is possible to obtain a family of differentiable equations whose only solutions are this series. It is appropriate to denote the set of lattice sites $\{(n, m); 0 \leq n \leq 2^{m-1} - 1, m \geq 1\}$ as Ω. Then the sought equations are

$$f\left(\frac{2n+1}{2^m}\right)-\frac{1}{2}\left\{f\left(\frac{n}{2^{m-1}}\right)+f\left(\frac{n+1}{2^{m-1}}\right)\right\}=c_m \tag{110}$$

for all $(n, m) \in \Omega$ and boundary conditions $f(0) = 0$ and $f(1) = c_0$.

Note that the left-hand side of (110) is essentially a central difference scheme for f. We can consider the modified equation (110) in the form

$$f\left(\frac{2n+1}{2^m}\right)=(1-\alpha)f\left(\frac{n}{2^{m-1}}\right)+\alpha f\left(\frac{n+1}{2^{m-1}}\right) \tag{111}$$

with the boundary conditions $f(0) = 0$ and $f(1) = 1$, where $0 < \alpha < 1$ is a constant. Then the only continuous solution (111) satisfies the following functional equation:

$$f(x)=\begin{cases}\alpha f(2x), & 0\leq x\leq \frac{1}{2},\\ (1-\alpha)f(2x-1)+\alpha, & \frac{1}{2}\leq x\leq 1.\end{cases} \tag{112}$$

Expression (112) is a special case of *the functional de Rham equation*, which proved the following

Theorem 6 (de Rum, 1957). *Suppose that F_0 and F_1 are contraction maps to R_n. Then the functional equation*

$$f(x)=\begin{cases}F_0(f(2x)), & 0\leq x\leq \frac{1}{2}\\ F_1(f(2x-1)), & \frac{1}{2}\leq x\leq 1\end{cases} \tag{113}$$

has a unique continuous solution if and only if $F_0(p_1) = F_1(p_0)$, where p_0 and p_1 are the only fixed points for F_0 and F_1, respectively.

Moreover, de Rham showed that a solution $L(\alpha, x)$ of the form (112) is strictly monotonically increasing, and its derivative vanishes almost everywhere if $\alpha \neq 1/2$.

Functions of this kind are known as *the Lebesgue singular functions $f_a(x)$*. From (112) for the values $\alpha = \beta = a$ and $g(x) = \cos\pi x$, the Weierstrass function is obtained; for $\alpha = \beta = 1/2$ and $g(x) = |x - [x + 1/2]|$ – Takagi function; for $\beta = 1$

– α, $g(x) = \alpha\theta(x - 1/2)$ is the singular Lebesgue function $f_\alpha(x)$, when θ is a step function.

For the solution of $L(\alpha, x)$, the following expression was also obtained:

$$L(\alpha,x) = x + \left(\alpha - \frac{1}{2}\right)\sum_{n\geq 0}\sum_{p=0}^{2^n-1}\alpha^{n-m(p)}(1-\alpha)^{m(p)}S_{p,n}(x), \quad (114)$$

where $m(p) = p - \sum_{n\geq 1}[p/2^n]$ and

$$S_{p,n}(x) = 2^n\left\{\left|x - \frac{p}{2^n}\right| + \left|x - \frac{p+1}{2^n}\right| - \left|2x - \frac{2p+1}{2^n}\right|\right\}. \quad (115)$$

From formula (114), one can obtain the exact interdependence between the Takagi function (79) and the solution to equation (112) in the form:

$$\frac{\partial}{\partial\alpha}L\left(\frac{1}{2}, x\right) = 2T(x). \quad (116)$$

Expression (114) is also applicable for the complex parameter $\alpha \in \{z; |z| < 1, |1-z| < 1\}$ and gives a continuous solution for (112). In particular

$$L\left(\frac{1}{2} + \frac{i}{2}, x\right) = x + \sum_{n\geq 0} 2^{-(n/2)-1}\sum_{p=0}^{2^n-1}S_{p,n}(x)\exp\left[\frac{\pi i}{4}(n+2-2m(p))\right] \quad (117)$$

defines the fractal curve, which was investigated by P. Levy in 1938.

De Ram showed in 1957 that the solution of the conjugate equation (112), i.e.

$$f(x) = \begin{cases}\alpha\overline{f(2x)} \\ (1-\alpha)\overline{f(2x-1)} + \alpha \quad 1/2 \leq x \leq 1.\end{cases} \quad (118)$$

is the Koch curve if $\alpha = 1/2 + (\sqrt{3}/6)i$ and the close-packed Polya curve if $\alpha = 1/2 + i/2$. The corresponding difference equations for (118) are a special case of the system written in the form:

$$\begin{cases} R\left(\frac{4n+1}{2^{m+1}}\right) = (1-\lambda_m)R\left(\frac{n}{2^{m-1}}\right) + \lambda_m R\left(\frac{n+1}{2^{m-1}}\right) \\ R\left(\frac{4n+3}{2^{m+1}}\right) = \mu_m R\left(\frac{n}{2^{m-1}}\right) + (1-\mu_m)R\left(\frac{n+1}{2^{m-1}}\right)\end{cases}, \quad (119)$$

for all $(n,m) \in \Omega$ and boundary conditions $R(0) = 0$ and $R(1) = 1$, $R(1/2) = \alpha$, where $0 < \lambda_m \leq \mu_m < 1$ and $m \geq 1$ are constants. Indeed, under the condition $\lambda_m = |\alpha|^2$ and $\mu_m = 1 - |1-\alpha|^2$, the continuous solution (119) also satisfies (118). Equations (119) have a unique continuous solution if $0 < \inf_{n\geq 1}\lambda_n \leq \sup_{n\geq 1}\mu_n < 1$.

It is clear that the curve $R(I)$ lies in a triangle with vertices 0, 1 and α. In the case $\lambda_n < \mu_n$ for $n \geq 1$ the curve $R(I)$ becomes the Jordan curve, and for $\lambda_n = \mu_n$ and $n \geq 1$ it becomes the Peano curve. The two-dimensional Lebesgue dimension of the curve $R(I)$ is given by the expression

$$\left[|\mathrm{Im}\,\alpha|\prod_{n\geq 1}(1-\lambda_n-\mu_n)\right]/2. \quad (120)$$

Thus, with an appropriate choice of $\{\lambda_n\}$ and $\{\mu_n\}$, the positive domain of the Jordan curve can be regarded as the only solution to (119).

2.13. Theorems on the construction of fractal sets

Two methods are widely used to construct self-similar fractal sets in the space R^p [58,77,110]. The first, adopted by Decking in 1982, uses endomorphisms of symbols in free groups, and the second, Hutchinson's method, is based on the use of a system of iterative functions, i.e. on the set of contraction maps. It should be borne in mind that the result of using iterative functions (attractor) is not always a fractal. In general, it can be any compact.

As noted above, the mapping $F: R^p \to R^p$ will be *contracting* if there is a constant $\lambda \in (0, 1)$ for which $\|F(x) - F(y)\| \leq \lambda\|x - y\|$ for all $x, y \in R^p$. The smallest λ is *the Lipschitz constant* F and is denoted by Lip(F). The only *fixed point* F is denoted here by the symbol Fix(F).

Then the following is true

Definition 1.1 (Hutchinson). *A non-empty subset* X *from* $\mathbf{R}^p$ *is invariant under the set m contracting mappings* $F_1, F_2, \ldots, F_m$ *if* X *satisfies the equality*

$$X = F_1(X) \cup F_2(X) \cup \ldots \cup F_m(X). \quad (121)$$

For the set of contracting mappings $F_1, \ldots, F_m$, one can define the mapping

$$\Phi(X) = F_1(X) \cup F_2(X) \cup \ldots \cup F_m(X) \quad (122)$$

for an arbitrary subset X from $\mathbf{R}^p$. Obviously, the sequence (122) converges to the fixed point Φ. It is also necessary to pay attention to the following result.

Theorem 7 (Williams, Hutchinson). *For the set of contracting images $F_1, ..., F_m$, there is a unique nonempty compact invariant set K. For an arbitrary nonempty compact subset X from* $\mathbf{R}^p$, *the system $\Phi^n(X)$ converges in the Hausdorff metric to K as $n \to \infty$.*

Also studied was the modification (121) of the inhomogeneous form

$$X = \Phi(X) \cup V = F_1(X) \cup ... \cup F_m(X) \cup V, \quad (123)$$

where V is a given compact subset of $\mathbf{R}^p$. Moreover, it was shown that there is a unique non-empty compact solution X satisfying (123). The following result is obtained.

Theorem 8 (Hata). *Suppose that $F_1, ..., F_m$ are continuous mappings such that the set $\bigcup_{n\geq 0}\Phi^n(X)$ is a precompactum for any compact set X. Then the following statements (a) and (b) are equivalent:*

(a) - there is a unique solution (123) for any compact set V;

(b) - Φ has a unique fixed point.

From the definition of the Hausdorff dimension of an invariant set, we have the following theorem.

Theorem 9 (Marion, Hutchinson). *Suppose that each contracting map F_j, $1 \leq j \leq m$, is a composition of stretch, rotate, translate, and flip operations. Suppose further that there is an open set U satisfying $\Phi(U) \subset U$ and $F_i(U) \cap F_j(U) = \varnothing$, for $i \neq j$. Then s, the dimensional Hausdorff measure of the invariant set K, is finite and positive, that is, $dim_H(K) = s$, where s is defined by $Lip(F_1)^s + \ldots + Lip(F_m)^s = 1$.*

For connected invariant sets, we have the following theorems.

Theorem 10 (Williams). *Let $Lip(F_1) + \ldots + Lip(F_m) < 1$, and each F_j is injective. Then K is completely disconnected and perfect.*

Recall that an *injective mapping* (*injection*) of a set A into a set B is a one-to-one mapping $f: A \to B$.

To study the connectedness of invariant sets, we can introduce the *structure matrix* $M_K = (m_{ij})$ of the set K in the form:

$$m_{ij} = \begin{cases} 1 & if\ F_I(K) \bigcap F_J(K) = \varnothing \\ 0 & in\ all\ other\ cases. \end{cases} \quad (124)$$

Then there is

Theorem 11 (Hata). *An invariant set K is connected if and only if its structure matrix M_K is irreducible. Moreover, if K is connected, it is also a locally connected continuum and path-connected.*

If two contracting mappings F_1 and F_2 satisfy the equality $F_1(\mathrm{Fix}(F_2)) = F_2(\mathrm{Fix}(F_1))$, then we can introduce a parametrization of the invariant set K using Theorem 6. Indeed, let $f(x)$ be a continuous solution to (113). Then

$$f(I) = f\left(\left[0, \frac{1}{2}\right]\right) \bigcup f\left(\left[\frac{1}{2}, 1\right]\right) = \\ = F_1(f(I)) \bigcup F_2(f(I)). \quad (125)$$

Therefore, $f(I)$ is a compact invariant set for F_1 and F_2, so $K = f(I)$, as required. Taking this into account, we have proved the following statement.

Theorem 12 (Hata). *Let f(x) be a continuous solution to (113). Then,*

(a) – if $\mathrm{Lip}(F_1)\cdot\mathrm{Lip}(F_2) < 1/4$, *then the Frechet derivative f vanishes almost everywhere;*

(b) – if each F_j is a homeomorphism and $\mathrm{Lip}(F_1^{-1})\cdot\mathrm{Lip}(F_2^{-1}) < 4$, *then f is not Frechet differentiable almost anywhere; moreover, if* $\mathrm{Lip}(F_j^{-1}) < 2$, *for $j = 1,2$, then f is nowhere differentiable.*

Note that the above results generalize Lax's theorem [110]. With such a parametrization, it is easy to obtain the well-known classical Peano curve, constructed by him in 1890, by Hilbert in 1891, and by Polya in 1913, using certain affine transformations in the space $\mathbf{R}^2$.

In conclusion, we note that, despite a significant number of works on nondifferentiable (fractal) functions and corresponding sets and mappings, it is too early to talk about the creation of their modern integral theory. Moreover, interest in them is currently growing significantly.

2.14. Classic fractal curves and sets

We begin our consideration with the *Cantor set* ("*Cantor dust*"), named after G. Cantor, who discovered it in 1883. The construction of the classical Cantor dust (**Fig. 6**) begins with the removal of the middle part of the segment, i.e.

open interval (1/3, 2/3). This is the first step in the iterative procedure. In the next and all other steps, we delete the middle third of all segments of the current level. The limit set C, which represents the intersection of all sets C_i, $I = 0, 1, 2, ...$, is called the classical Cantor dust.

Cantor dust is a fractal of dimension $D = \ln 2/\ln 3 \approx 0.6309$. The sum of all the lengths of the intervals removed when constructing the set C is exactly 1. The total "length", or measure, of the remaining set is equal to zero. However, the remaining "dust" still contains innumerable dots. Formally, a Cantor set is defined as completely discontinuous, closed, and perfect. It can be used to construct a continuous fractal function by integrating a suitable distribution function given on a Cantor set. Then we get a fractal function called "devil's ladder". In particular, such functions play a very noticeable role in the theory of oscillations in describing frequency synchronization, when the so-called "Arnold tongues" arise.

Another completely non-intuitive consequence of Cantor sets is *the equivalence of two-dimensional domains and one-dimensional lines*. Two sets are equivalent if there is a one-to-one correspondence between them. For example, the unit square and the unit line segment are equivalent: each point of the unit square corresponds to one point of the unit line and vice versa. In this regard, Kantor wrote: "*I see, but I do not believe.*"

Who would have thought that such contradicting common sense mathematical constructions, invented only to convince

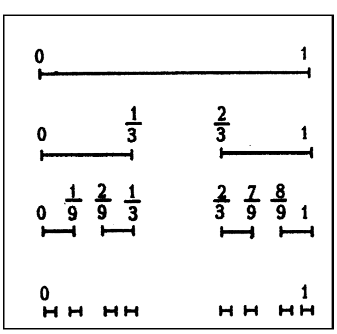


Fig. 6. *Construction of the Cantor set.*

skeptics of the possibility of the existence of uncountable sets of zero measure, would become one of the central concepts and find practical application? Meanwhile, Cantor sets later turned into almost ideal models for many branches of modern natural science - from strange attractors to the distribution of galaxies in the Universe. It is appropriate in this connection to quote the statement of W. Hilbert: "*No one can expel us from the paradise that Cantor created for us.*"

Fractal functions are *non-differentiable functions*. They originated over a hundred years ago. The scientific community of the past called them "monsters" (often adding the epithet "pathological"), of interest only to those specialists who are characterized by mathematical quirks, but not to professional scientists. This was perceived as the destruction of mathematics: S. Hermite wrote to T. Stieltjes in 1893: "*I turn away with horror and disgust from this growing ulcer of functions that have no derivatives*".

However, the time of mathematicians' faith in the inseparability of the connection between continuous functions and their differentiability has expired (see paragraphs 6-10). From the standpoint of modern science, *a function without a derivative is not at all an abstract concept,* but the trajectory of a Brownian particle. In the arsenal of mathematics, there was also an analytical apparatus for describing such objects. The place of the usual dimension was taken by *the Hausdorff dimension*, and the place of derivatives was taken by *the fractional derivative* (see item 4). In 1906, J. Perrin stated that "*curves without tangents are a general rule, and smooth curves, such as a circle, are an interesting but very special case.*"

The curve in **Fig. 7** was originally described by Helge von Koch in 1904. Every third of the curve is constructed iteratively, starting with a line segment (initiator). Let's remove the middle third and add two new line segments. The result of this construction is called a generator. The length of the generator is 4/3 of the length of the initiator. We repeat this procedure many

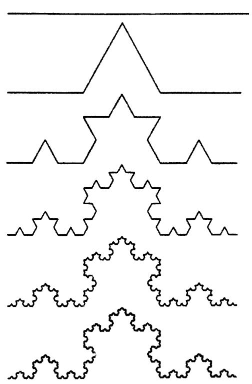


Fig. 7. *Construction of the Koch curve.*

times, at each step replacing the middle third with two new segments. In the limiting case, the nowhere nondifferentiable Koch curve is a line of infinite length with fractal dimension D = ln4/ln3 ≈ 1.2618.

Applying the von Koch generator to an equilateral triangle, through an infinite number of iterations, we arrive at the von Koch snowflake (**Fig. 8**). In the limit, this curve also has infinite length, limiting an area equal to 8/5 of the area of the original triangle. The von Koch snowflake does not cross itself anywhere. If the triangles are built inward, and not outward, then a curve is obtained - an anti-snowflake, its perimeter is infinite, and it limits an infinite set of disconnected regions, with a total area of 2/5 of the area of the original triangle.

Deterministic fractals, called the Sierpinski napkin and carpet or Sierpinski curves (1915), are obtained by sequentially cutting out triangles (**Fig. 9***a*) or squares (Fig. 9*b*).

In the limit, at the Sierpinski napkin, the black areas disappear, and the full perimeter of the holes tends to infinity. Thus, in the process of constructing the napkin, an area exactly equal to the area of the original triangle will be excluded. The fractal dimension of the Sierpinski napkin is D = ln3/ln2 ≈ 1.5849. One more property of the

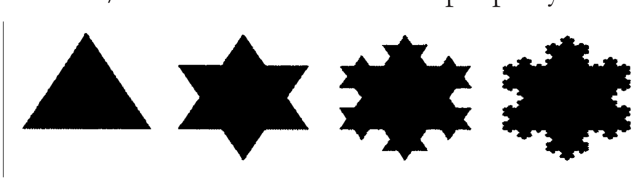


Fig. 8. *Construction of the Koch snowflake.*

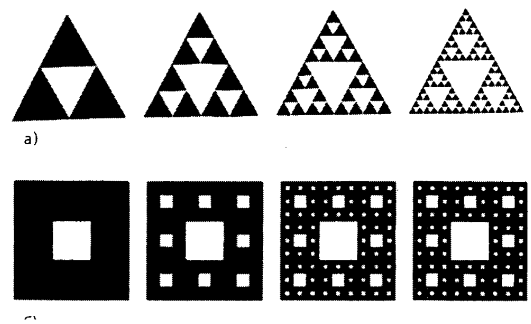


Fig. 9. *Construction of napkins (a) and carpet (b) Sierpinski.*

Sierpinski napkin can be noted. For Euclidean bodies in *n*-dimensional space, the volume V is proportional to R^n, where R is a certain characteristic size of the body. The surface area S changes in proportion to R^{n-1}, therefore $S \propto V^{(n-1)/n}$. However, for a Sierpinski napkin, the area and length of the edges are proportional to each other: $S \propto V$!

The Sierpinski napkin combines self-similarity with rotation symmetry. The shape of the Sierpinski napkin does not change when it is rotated through an angle multiple of 120°. Symmetry data (infinite scaling and rotation through a finite angle) are observed in well-known paintings by Maurice Escher. Two-dimensional and three-dimensional analogs of the Sierpinski napkin model many natural and man-made structures (for example, the Eiffel Tower in Paris). For the Sierpinski carpet, D = ln8/ln3 ≈ 1.8928, i.e. it is, in a sense, less leaky. The Sierpinski carpet is an analogue of a Cantor set on a square. The Sierpinski curve consists entirely of branch points alone.

The Koch snowflake and other fractal curves on the plane are united by the fact that their dimension lies in the range $1 < D < 2$. The question arises, is there a curve of dimension 2? This question was resolved by *D*. Peano in 1890. The Piano curve in the limit fills the square so densely that its D = 2. At the same time, the Peano curve is a graph of a continuous function. Nevertheless, at no point can a tangent be drawn to it, since at any moment in time we do not know the direction in which the point is moving. The concept of the Peano curve is not intuitive, but originally arose from purely analytical reasoning.

W. Hilbert in 1891 proposed a simple method for constructing a Peano curve with two endpoints. In **Fig. 10** shows the first four steps of his recursive procedure. At the limit, the curve starts and ends at the top vertices of the square.

A variant of constructing a closed Peano curve belongs to Sierpinsky and is shown in **Fig. 11**. In each of the options, the limit curve has an infinite length and completely fills the square. The approximate curves limit the areas, which tend to 5/12 in the limit, but for the graph of the limiting function, the difference between the inner and outer parts of the square loses its meaning.

Peano curves easily generalize to higher topological dimensions and can fill cubes and hypercubes. Hilbert's constructions have found interesting applications in information theory in Gray codes. Several ways to scan an image in television use the Hilbert algorithm. The point is that points adjacent in time along the "Hilbert sweep" are adjacent in space and on the scanned image, which simplifies its processing.

The Sierpinski carpet satisfies Uryson's definition of a line. Therefore, any Cantor curve, being homeomorphic to a subset of the Sierpinski carpet, is also one-dimensional and is a line in the sense of P.S. Uryson. Conversely, if a flat continuum is one-dimensional, then it will be a Cantor curve.

There are lines that are not homeomorphic to any subset of the plane. At the same time, by Menger's *theorem*, any line is homeomorphic

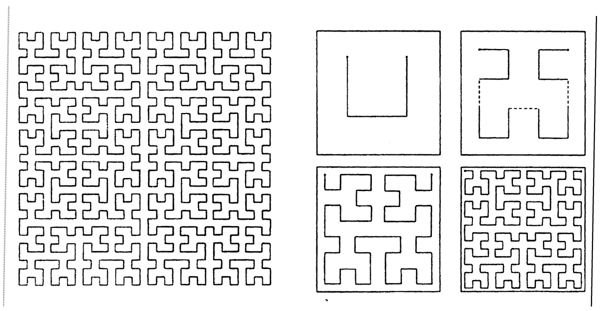


Fig. 10. *Peano's curve constructed by the Hilbert algorithm (left) and the first four iterations (right).*

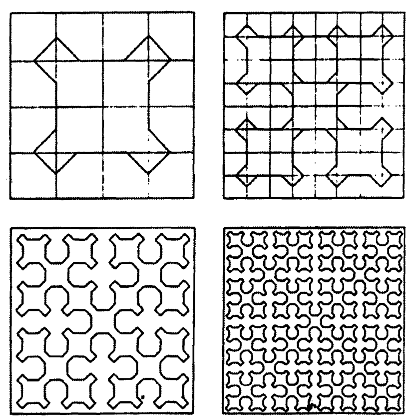


Fig. 11. *The closed Peano curve constructed by Sierpinsky.*

to some subset of three-dimensional Euclidean space. More general is the Nöbeling-Pontryagin *theorem*. In 1926, Menger constructed in $\mathfrak{R}^3$ a one-dimensional continuum M_1^3, which topologically contains any line. This continuum is called the universal Menger curve.

The construction of the universal Menger curve M_1^3 is as follows. A cube I^3 with a single edge is divided by planes parallel to its edges into 27 equal cubes with an edge 1/3. Then the inner cube and 6 adjacent cubes (cubes of the first rank) are removed. The remaining set K_1 consists of 20 cubes of the first rank. Proceeding also with each of the cubes of the first rank, we obtain the continuum K_2, consisting of 400 cubes of the second rank. In the process of infinite construction, we have a decreasing sequence of continua $I^3 = K_0 \supset K_1 \supset K_2 ...$, the intersection of which is a one-dimensional continuum I_1^3. The first steps in constructing the Menger curve are shown in **Fig. 12.**

We can say that the abstract constructions of Cantor and Peano have provided us with models *of reality much more realistic* than the entire Euclidean geometry of integer exponents and smooth forms.

There are curves in which, in contrast to Peano's original construction, there are no self-contact points. One example of this kind is the Gosper curve. The initiator for it is a segment of unit length, and the generator is shown in the upper right in **Fig. 13**. It consists of 7 sections,

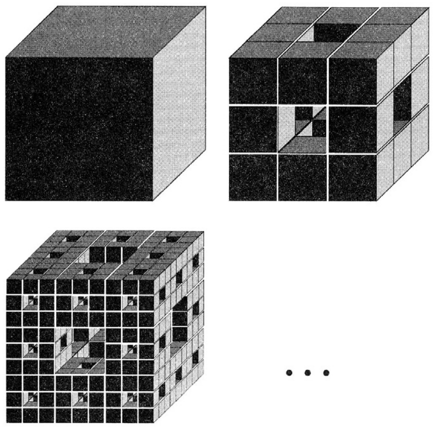


Fig. 12. *Construction of the universal Menger curve.*

each $1/\sqrt{7}$ long The dotted line denotes a triangular lattice, which serves as a kind of generatrix for this generator. The next three steps in the build process are shown in Fig. 13 below.

The dimension of the Gosper curve is $D = 2$. A distinctive feature of this curve is that the boundary of the region called the "Gosper island", which it fills in the limit, is itself fractal with $D \approx 1.1291$. These islands can be used for continuous plane coverage as they fit perfectly together. Moreover, seven such islands, docked so that one is in the center and six around it, again form the island of Gosper, three times larger. Of regular polygons, only a square has this property.

Let's give another example of the Peano curve – a fractal called "Harter-Hatway's dragon". The first four steps of its construction are shown at the top of **Fig. 14**. Each of the segments in the next step is bent at right angles. The fold direction is alternating. After each step, the number of segments is doubled, and the length of each

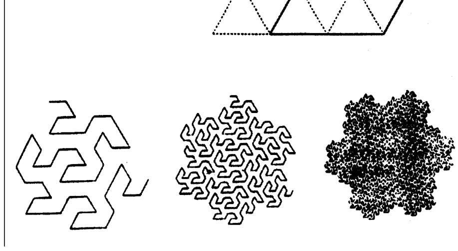


Fig. 13. *Generator of the Gosper curve and its iterations.*

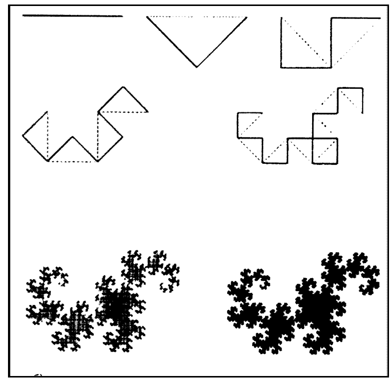


Fig. 14. *The first four stages of the construction of the "Harter-Hatway dragon" (a) and its subsequent outlines.*

segment decreases in $\sqrt{2}$. Therefore, in the limit $D = 2$. The shape of the resulting unusual figure is shown at the bottom of Fig. 14 for 12th and 16th dragon generations. The dragon curve is self-similar.

Consecutive central folds precisely fit a logarithmic spiral, which itself is one of the main smooth self-similar objects, and has practical application in the design of broadband antennas for a variety of radio systems [57,58,62,65]. Nature also uses the self-similarity of the logarithmic spiral, for example, the self-similar shell of the multichamber mollusk Nautilus.

It is amazing that a fairly simple algorithm leads to such an unusual figure as a dragon. The biological subtext inherent in the name of the curve makes one wonder: is it not encoded in genes in a similar way information about the shapes and sizes of existing living organisms?

2.15. Methods for the synthesis of fractals and fractal sets on the complex plane

When modeling deterministic fractals, special methods are used, such as systems of L-functions and systems of iterated functions (IFS) [28, 29, 57, 58, 62, 115].

The concept of L-systems appeared in 1968 thanks to A. Lindenmayer. L-systems were first introduced in the study of formal languages and were also used in biological breeding models. For the graphical implementation of L-systems, *turtle-graphics* are used as an inference subsystem. A deterministic L-system formally consists of an alphabet, an initialization word called an axiom or initiator, and a set of generating rules (generator).

One of the deepest and most remarkable advances in the construction of fractals is the *system of iterated functions*. The mathematical foundations were developed by J. Hutchins (1981), and the method itself became widely known thanks to M. Barkley (1988). A system of iterated functions is a collection of *affine transformations*. As you know, affine transformations include scaling, rotation, and parallel translation. There are two approaches to implementing IFS: deterministic and randomized. *The deterministic algorithm* produces attractive images, but requires processing large arrays of zeros and ones. In *a randomized algorithm*, the initial set contains only one point. At each step, only one affine transformation is used from the entire set of transformations that define the IFS. This transformation is chosen at random.

Nonlinear algorithms for constructing fractals use iterations on a complex plane of the form $z_{n+1} = z_n^2 + c,$ where *c* is some complex constant that is a control parameter. The apparent simplicity of this process is in no way comparable to the stunning beauty and variety of those fractal structures that arise in this process. In 1879 Sir Arthur Cayley posed the problem of iterating complex functions. The theory of iterations on the complex plane was described in 1918 by G. Julia (1893-1978), who was then in the hospital after being wounded at the front during the First World War. Both his work and the work (1919) of his contemporary and rival P. Fatou (1878-1929), were soon consigned to oblivion. As noted in [28], the most significant and impressive contribution was made by Fatou himself, but Julia made him strong competition and had some advantages associated with his status as a wounded war hero. In 1918, Julia received the "Grand Prix of Mathematical Sciences" from the Paris Academy of Sciences for his work.

The studies of P. Montel, D. Sullivan, B. Mandelbrot, J. Milnor and others have again drawn attention to their theory. The intellectual achievements of G. Julia and P. Fatou are also notable for the fact that they had to rely entirely on the imagination. Computers have made visible what could not be depicted during the years of the creation of this theory. Visual computer results exceeded all expectations.

Julia sets are fractal boundaries arising in the process of iterating a quadratic complex transformation that preserves angles, i.e. *conformal transformation*. The variety of border shapes depends only on the control parameter *c*. For some values of *c*, the Julia sets are connected (**Fig. 15**), and for other values, they are completely disconnected and represent dusty Cantor sets (*Fatou dust* - **Fig. 16**).

It turned out that absolutely all values of the parameter *c* for which the Julia set is connected belong to *the Mandelbrot set* (*M-set*), discovered in 1980. The Mandelbrot set is shown in **Fig. 17** as the blacked-out portion of the complex *C*-plane.

From an arbitrary point of the set *M* one can get to any other point without leaving the set *M*, i.e. the Mandelbrot set is connected (Douady and Hubbard 1982). This is not just a bizarre form that seems beautiful to someone, but ugly to someone; it embodies, more general than Feigenbaum's universality, *the principle of the transition from order to chaos*. The subtle mathematical web of the Mandelbrot set continues to awe even seasoned professionals. The complexity of the *M*-set is a reminder that *complexity in many natural phenomena can be a consequence of simple laws*.

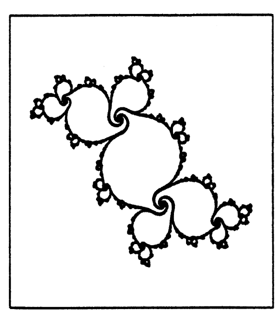


Fig. 15. *Julia set.*

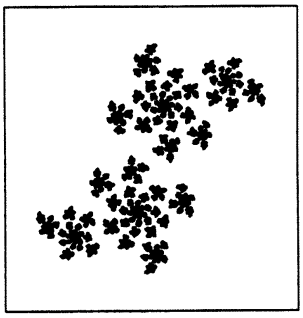


Fig. 16. *Fatou set.*

If we take the values of the constant *c* outside *M*, then the only attractor will be infinity, i.e. the point representing the iteration process goes to infinity. Then Julia's multitude disintegrates into Fatou dust. This dust becomes finer and finer with distance *c* from *M*. If point *c* is near the boundary of *M*, then the dust forms fascinating figures, examples of which are shown in Fig. 17 and **Fig. 18**. These figures are *always fractal, self-similar and carry chaotic dynamics.*

The most remarkable feature of the Mandelbrot set is that it serves as an infinitely efficient storage of images (Tang Lei, 1984). With an increase in the Mandelbrot set, in the vicinity of its boundary point *c*, forms appear that are Julia sets.

All fractals discussed above were *deterministic.* The construction of *random* fractals is not reduced to random perturbations of deterministic fractals. On the contrary, a random nature is inherent in them initially, which is associated with random processes. The main model for constructing random fractals is *fractal Brownian motion*. The existence of a fractal Brownian motion was proved by B. Mandelbrot and Van Ness in 1968.

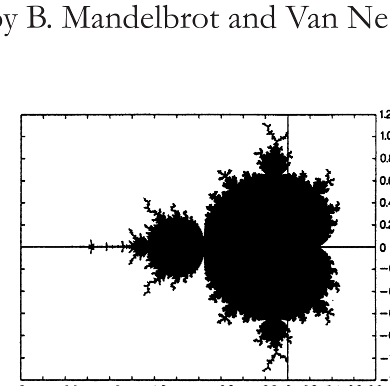


Fig. 17. Mandelbrot set.

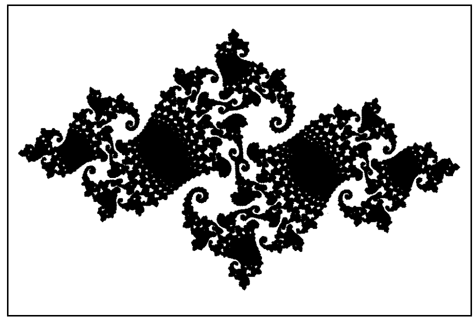


Fig. 18. *Fatou set.*

This process was implicitly considered by A.N. Kolmogorov in 1940.

The *method of random perturbations* is sometimes used. For a *randomized Koch snowflake*, equilateral triangles are added, randomly facing both inward and outward. In *a randomized Sierpinski napkin*, any of the triangles are randomly removed during construction. Sometimes, randomization of the lengths of the intervals removed when constructing the Cantor set is used. Summarizing what has been said, it can be noted that *random fractals are combinations of generative rules chosen at random on different scales*. In this case, in the iterative procedure, you can randomly change its parameters.

2.16. Hurst exponent of random processes

In fractal processing of samples $\xi(t)$ of a random process and time series, *the normalized range method* or *the Hurst method* [57,58,62,89,115] is currently often used. In this case, such processes are characterized by *the Hurst exponent* or *the codimension H*. To calculate the Hurst exponent *H* of a one-dimensional sample, it is necessary to calculate its normalized range *R*/*S*. For all kinds of random processes, this value obeys the following empirical relationship:

$$R/S = (\tau/2)^H. \tag{126}$$

In formula (126), the expression

$$R(\tau) = \max_{1\le t\le\tau} X(t,\tau) - \min_{1\le t\le\tau} X(t,\tau), \tag{127}$$

the maximum range of amplitudes of a random process in the sample under consideration,

$$X(t,\tau) = \sum_{u=1}^{t} \{\xi(\mathrm{u}) - \langle\xi\rangle_\tau\} \tag{128}$$

deviation of $\xi(u)$ from the mean,

$$\langle \xi \rangle_\tau = \frac{1}{\tau}\sum_{t=1}^{\tau}\xi(t) \tag{129}$$

average value over the interval τ,

$$S = \left\{\frac{1}{\tau}\sum_{t=1}^{\tau}\left[\xi(t)-\langle \xi \rangle_\tau\right]^2\right\}^{1/2} \tag{130}$$

root-mean-square deviation, t – discrete time with integer values, τ – duration of the considered time interval.

It should be noted that the Hirst method is an extremely robust method. It does not have an initial assumption of *Gaussian distributions*.

From relation (126), the value of the Hurst exponent H is determined by logarithm. For a one-dimensional reflected signal, the fractal dimension D, which *characterizes its structural properties*, under the condition $0 \le H \le 1$, is related to the Hurst exponent as follows:

$$D = 2 - H. \tag{131}$$

In the case of a two-dimensional process (image) with parameter H, relation (131), which determines the fractal dimension D, must be written in the form:

$$D = 3 - H. \tag{132}$$

It is well known [57,58,62,89,115] that the case

$$1/2 < H < 1 \tag{133}$$

corresponds to *a persistent process* (a process that preserves the observed tendency of an increase or decrease in instantaneous amplitudes in a sample, i.e., a process with memory). At the same time, trends in the process under study are obvious. This is true on average and for arbitrarily large time intervals t, when the time series becomes less noisy.

Happening

$$0 < H < 1/2 \tag{134}$$

corresponds to *an antipersistent process* (in this case, an increase in the amplitudes of the signal envelope in the "past" means a decrease in the "future", and vice versa). The antipersistent H value characterizes a system that is more susceptible to change. This type of system is often referred to as "Return to Mean".

The value $H = ½$ corresponds to *the classical Brownian motion*, which is a Markov process. The structural functions of a random process are also used to estimate the Hurst parameter. In practice, it is believed that the Hurst exponent can be estimated fairly accurately using a sample of about 2500 measurements [57,58].

In the case of fractal analysis of signals, in the general case, it is necessary to construct graphs of the dependence of the variance of the signal or its structure function on a double logarithmic scale. If the obtained dependence is sufficiently well approximated by some "straight line" on a large number of time scales, then the tangent of its slope can be used to find the value $2H$ (when analyzing the time course of the signal dispersion) or the value H (when the structure function is analyzed). The linear section of the obtained experimental "straight line" will determine the scaling region of the process under study.

For fractal processes describing one-dimensional *generalized Brownian motion* with codimension $0 < H < 1$, the spectral power density $G(f)$ has a fractal form [57,58]:

$$G(f) = \frac{1}{f^{\alpha}},\ a = 2H+1. \tag{135}$$

The use of the H exponent in radiophysical problems is briefly presented in Section 2. These issues are related to the circle of general issues of the evolution of open radiophysical systems with a change in external parameters and the appearance of a chaos regime and bifurcation points, i.e., to solving urgent problems of an adaptive scheme of a fractal detector radar signals, which is also shown in clause 2.

2.17. Fox functions and processes in fractal environments

The application of the apparatus of Fox functions [116] when considering the processes of relaxation and diffusion in media with fractal dimension, characterized by an equation of diffusion type with fractional derivatives with respect to coordinates and time, will be demonstrated on the basis of the results obtained in [22]. Because of fractality – in contrast to the standard diffusion equation, when the particle flux and $j \sim \partial\rho/\partial t$, $j \sim \partial^2\rho/\partial x^2$ – is violated due to self-similarity, the locality of such connections. The flow value begins to depend on the prehistory of the process – the concentration values at earlier points in time:

$$j(t) \approx \frac{\partial}{\partial t}\int_0^t \rho(\tau)K(t,\tau)d\tau, \tag{136}$$

those. diffusion and relaxation processes become non-Debye.

In formula (136), the kernel $K(t, \tau)$ includes the fractal dimension D of the medium and in the stationary mode depends on the difference of the arguments. At the same time, $K(t, \tau)$, when replacing a fractal medium with an ordinary one, must satisfy the standard diffusion equation. The simplest such kernel is the power function $K(t - \tau) = (t - \tau)^{-\nu(D)}$ with the exponent depending on the fractal dimension of the diffusion space D. In this case, the right-hand side of (136) coincides in structure with the definition of the fractional derivative Riemann-Liouville (36) of order $0 < \nu < 1$, i.e. $j(x,t) \sim \partial^\nu \rho(x,t)/\partial t^\nu$. At the same time, due to the complexity and entanglement of the particle trajectories, the derivative with respect to the coordinate (gradient) becomes fractal and $j \sim \partial^{2\nu}\rho/\partial t^{2\nu}$. The equations of non-Debye diffusion and relaxation take the following form, respectively:

$$\frac{\partial^\nu}{\partial t^\nu}\rho = \tilde{D}\frac{\partial^{2\gamma}}{\partial x^{2\gamma}}\rho,\ \ \frac{\partial^\nu}{\partial t^{2\gamma}}\rho = -\frac{1}{\tau^\nu}\rho, \tag{137}$$

where $0 < \nu < 1$, $1 < 2\gamma \le 2$ [57,58].

Diffusion-relaxation processes investigated in [22] are described in the one-dimensional case by the equation

$$\frac{\partial^\nu}{\partial t^\nu}\rho = \tilde{D}\frac{\partial^{2\gamma}}{\partial x^{2\gamma}}\rho - \frac{1}{\tau^\nu}\rho, \tag{138}$$

and for their solution the above mathematical apparatus of Fox functions is used.

Acting on the right and left sides of Eq. (138) by the fractional integral operator (30) in the form

$${}_0D_t^{-\nu} g = \frac{1}{\Gamma(\nu)}\int_0^t dt' \frac{g(t')}{(t-t')^{1-\nu}}, \tag{139}$$

we get

$$\rho(x,t) - \rho_0(x) = {}_0D_t^{-\nu}\left\{\left[\frac{\partial^{2\gamma}}{\partial x^{2\gamma}}\tilde{D} - \frac{1}{\tau^\nu}\right]\rho(x,t)\right\} \tag{140}$$

under the initial conditions $\rho(x,t)|_{t=0} = \rho_0(x)$. Fractional differentiation (140) using the operator ${}_0D_t^\nu$ under the condition ${}_0D_t^\nu\,{}_0D_t^\nu = 1$ gives

$${}_0D_t^\nu\{\rho(x,t) - \rho_0(x)\} = \left[\frac{\partial^{2\gamma}}{\partial x^{2\gamma}}\tilde{D} - \frac{1}{\tau^\nu}\right]\rho(x,t). \tag{141}$$

Next, you need to use the Fourier transform in relation to the spatial coordinate and the Laplace transform $\tilde{\rho}(k,p) = \int_0^\infty \rho(k,t)\exp(-pt)dt$ in time. The Fourier amplitude $\tilde{\rho}(k,p)$ satisfies the equation

$$\begin{aligned} &{}_0D_t^\nu\{\hat{\rho}(k,t) - \rho_0(x)\} = \\ &= \left[\tilde{D}(ik)^{2\gamma} - \frac{1}{\tau^\nu}\right]\hat{\rho}(k,t) = -\frac{1}{T^\nu}\hat{\rho}(k,t). \end{aligned} \tag{142}$$

The action of the fractional time derivative on a time-independent function $\rho_0(x)$ is not equal to zero: ${}_0D_t^\nu\rho_0 = \rho_0 t^{-\nu}/\Gamma(1-\nu)$.. For the image $\tilde{\rho}(k,p)$, we obtain the equation

$$\tilde{\rho}(k,p) = \frac{p^{-1}}{1+(Tp)^{-\nu}}\hat{\rho}_0(k). \tag{143}$$

Next, we represent $\tilde{\rho}(k,t)$ through the Fox function:

$$\tilde{\rho}(k,p) = \frac{T(k)\hat{\rho}_0(k)}{\nu}H_{1,1}^{1,1}\left(T(k)p\left|\begin{matrix}1-1/\nu,1/\nu\\1-1/\nu,1/\nu\end{matrix}\right.\right), \tag{144}$$

and carry out the inverse Laplace transform

$$\hat{\rho}(k,t) = \frac{\rho_0(k)}{\nu}H_{1,2}^{1,1}\left(\frac{t}{T}\left|\begin{matrix}(0,1/\nu)\\(0,1/\nu),(0,1)\end{matrix}\right.\right). \tag{145}$$

The solution to equation (140) has the form

$$\begin{aligned} \rho(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\exp(ikx)\hat{\rho}_0(k)\times \\ &\times H_{1,2}^{1,1}\left(-t^\nu\left[(ik)^{2\gamma}\tilde{D} - \frac{1}{\tau^\nu}\right]\left|\begin{matrix}(0,1)\\(0,1),(0,\nu)\end{matrix}\right.\right). \end{aligned} \tag{146}$$

We represent the function $H_{1,2}^{1,1}$ in the form of a series and obtain the solution of equation (140) in the form

$$\rho(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp(ikx)\times$$
$$\times\hat{\rho}_0(k)\sum_{n=0}^{\infty}\frac{\left[\tilde{D}(ik)^{2\gamma}t^{v} - t^{v}\tau^{-v}\right]^{n}}{\Gamma(1+vn)}. \tag{147}$$

Integral (146) is calculated explicitly in a number of cases. For $1/\tau = 0$, formula (146) is reduced to the form

$$\rho(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp(ikx_1)\hat{\rho}_0(k)\times$$
$$\times H_{1,2}^{1,1}\left(-(ik)^{2\gamma}\left|\begin{matrix}(0,1)\\(0,1),(0,v)\end{matrix}\right.\right)(\tilde{D}t^{v})^{-1/2\gamma}. \tag{148}$$

For Fox functions, it is more convenient to use the sine (F_s) and cosine (F_c) of the Fourier transform. Consider a particular case $\hat{\rho}_0(k) = \rho_0 = const$, $\rho_0(x) = \rho_0\delta(x)$, and let $\gamma = 1 - \varepsilon$, $0 \le \varepsilon \le 0.5$. After a series of transformations [22], we obtain an exact solution of equation (140) with $1/\tau = 0$ and the initial condition $\rho_0(x,t)|_{t=0} = \rho_0\delta(x)$, at $0.5 < \gamma < 1$:

$$\rho(x,t) = \frac{\rho_0}{4\gamma(\tilde{D}t^{v})^{1/2\gamma}}\times$$
$$\times\left[\begin{aligned}&\frac{(-1)^{\varepsilon/2\gamma}}{x_1}H_{3,3}^{2,1}\left((-1)^{\varepsilon/2\gamma}x_1\left|\begin{matrix}(1,1/2\gamma),(1,v/2\gamma),(1,1/2)\\(1,1),(1,1/2\gamma),(1,1/2)\end{matrix}\right.\right)+\\&+\frac{(-1)^{\varepsilon/2\gamma}}{x_1}H_{3,3}^{2,1}\left((-1)^{-\varepsilon/2\gamma}x_1\left|\begin{matrix}(1,1/2\gamma),(1,v/2\gamma),(1,1/2)\\(1,1),(1,1/2\gamma),(1,1/2)\end{matrix}\right.\right)+\\&+\frac{(-1)^{-\varepsilon/2\gamma+1/2}}{x_1}H_{3,3}^{2,1}\left((-1)^{\varepsilon/2\gamma}x_1\left|\begin{matrix}(1,1/2\gamma),(1,v/2\gamma),(1/2,1/2)\\(1,1),(1,1/2\gamma),(1/2,1/2)\end{matrix}\right.\right)-\\&-\frac{(-1)^{\varepsilon/2\gamma+1/2}}{x_1}H_{3,3}^{2,1}\left((-1)^{-\varepsilon/2\gamma}x_1\left|\begin{matrix}(1,1/2\gamma),(1,v/2\gamma),(1/2,1/2)\\(1,1),(1,1/2\gamma),(1/2,1/2)\end{matrix}\right.\right)\end{aligned}\right]. \tag{149}$$

For $\gamma = v = 1$, after a series of transformations, the solution to Eq. (149) coincides with the well-known solution to the ordinary diffusion equation.

The asymptotic expansion for the function $\rho(x,t)$ as $t \to \infty$ allows us to write:

$$\rho(x,t) \approx \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{t}{\Gamma\left[1 - v(n+1)\right]}\times$$
$$\times\int_{-\infty}^{\infty}\frac{\hat{\rho}_0(k)\exp(ikx)}{\left[\tilde{D}(ik)^{2\gamma} - 1/\tau^{v}\right]^{n+1}}, t \to \infty, \tag{150}$$

For specific ones $\hat{\rho}_0(k)$, explicit expressions can be obtained (150). Let $\hat{\rho}_0(k) = \rho_0\delta(k), \tilde{D} = 0.$ Then

$$\rho(x,t) = \rho(t) =$$
$$= \frac{1}{2\pi}\rho_0\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\Gamma\left[1 - v(n+1)\right]}\left(\frac{t}{\tau}\right)^{-v(n+1)}, \quad t \to \infty, \tag{151}$$

and $\rho(t) \sim 1/t^{v}$.

For $\rho_0(k) = \rho_0 = const$ and $1/\tau^{v}$, we are dealing with pure diffusion. The solution is given by expression (149) and the diffusion displacement of the particle with time $x \sim t^{v/2\gamma}$. For $v = \gamma = 1$, we obtain the well-known relation $x^2 \sim t$. Approximate summation of the series in n in (147) for large t, which is carried out by preserving a small number of terms in the series in n and replacing the function $\Gamma(1 + vn)$ by the function $(1 + n)$ as $v \to 1$, allows one to obtain the asymptotics in t and for $\gamma \neq 1$.

2.18. Wave equation and fractal media

The results of studying a nonlinear equation of the generalized wave equation type in fractal space, a particular case of which are both the wave equation for a nonlinear medium and the nonlinear diffusion equation, are given in [23]. This equation describes the processes with the preservation of temporal and spatial memory in the form

$$D_{+,t}^{v}\rho(x,t) =$$
$$= \tilde{D}_0 D_{+,x}^{\gamma}\left[\rho^{\sigma}(x,t)D_{+,x}^{\gamma}\rho(x,t)\right], v \ge 0, \gamma \ge 0, \tag{152}$$

where $D_{+,t}^{v}$ and $D_{+,x}^{\gamma}$ are the fractional Riemann-Liouville derivatives with respect to time and coordinate, $x > 0$; $\tilde{D}_0$ = const.

This equation differs from the nonlinear diffusion equation [57,58] of the form

$$\frac{\partial}{\partial t}\rho(t,x) = \frac{\partial}{\partial x}\left[\tilde{D}_0\rho^{\sigma}(t,x)\frac{\partial}{\partial x}\rho(t,x)\right] \tag{153}$$

with autowave solution

$$\rho = \rho_0\left[v^{-1}(vt - x)\right]^{1/\sigma}, \quad 0 \le x \le vt \tag{154}$$

by replacing time and space derivatives with fractional derivatives, considered as generalized functions. In this case, the boundary and initial conditions of Eq. (153) are preserved for (152), and $\rho(x,t)$ is considered as a generalized function.

Equation (152) describes the propagation of electromagnetic waves in nonlinear fractal or weakly fractal (v ≈ 2, γ ≈ 1) media with an appropriate choice of values of v, γ, σ. With the addition of nonlinear terms in terms $\rho(x,t)$, Eqn (152) will describe both nonlinear diffusion and self-organization processes [57, 58].

3. APPLIED ASPECTS OF THE METHOD

3.1. Modern physical concept based on the theory of fractals and fractional operators

At present, there has been a noticeable increase in interest in understanding concepts such as *simplicity* and *complexity*, awareness of the various unique features of complex systems of animate and inanimate nature, including dissipativity, self-organization, fractality, scaling, heredity (non-Markovness), non-Gaussianity, linear and nonlinear responses to external disturbances. One of the main problems in analyzing signals produced by complex systems is the adequate parameterization of the contributions that characterize the components of the signals under study. Typically, the features of complex systems are manifested at different spatio-temporal scales. As is well known, stationary regimes and periodic motions have long been considered the only possible states. However, the discoveries of the second half of the 20th century radically changed our understanding of the nature of dynamic processes. Now we realize that our world is not only *nonlinear*, but also *fractal*. At present, the lack of traditional physical models is clearly felt.

Note that the main problems of radiophysics include the issues of radar detection of high-speed, stealthy and small-sized objects near the surface of the earth and the sea, as well as in meteorological precipitation, which is an extremely difficult task for high-speed targets and unpredictable trajectories [1,17,39,65]. In addition, interference from the sea surface and vegetation is of a non-stationary and multi-scale nature, especially at low grazing angles ϑ. Recently, more and more different types of unmanned aerial vehicles (UAVs) have appeared. Due to their small dimensions, as well as the use of plastics, fiberglass, polystyrene, even cardboard and other weakly reflecting electromagnetic waves in their designs, UAVs have a small effective reflectivity. The signal-to-noise ratio q_0^2 for the tasks listed above almost always fills the range of negative (in decibels) values, i.e. $q_0^2 < 0 \ldots 1$ dB.

As is well known, in experiments on the scattering of meter, decimeter, centimeter and millimeter waves, researchers faced questions of the legitimacy and applicability of Gaussian models. Soon, numerous artificial attempts began to create scattering models in order to increase the level of the "tails" of the probability distributions of the amplitudes of the reflected signals.

All this makes classical radar methods and detection algorithms difficult to apply, i.e. the use of energy detectors (when the likelihood ratio is determined exclusively and only by the energy of the received signal) becomes fundamentally impossible. The detection of low-contrast objects against the background of the above natural intense interference inevitably requires the introduction and calculation of some fundamentally new characteristic, which differs from the classical functionals associated with the energy of interference and signal, and is *determined solely by the topology and dimension of the received signal mixture with interference and noise.*

The application of the ideas of scale invariance - “scaling” - together with set theory, fractional dimension theory, fractional calculus, general topology, geometric measure theory, and dynamical systems theory opens up great potential and new perspectives in multidimensional signal processing and related scientific and technical fields. In other words, a complete description of the processes of modern signal and field processing is impossible using the approaches and formulas of only classical mathematics.

With the fractal-scaling approach, proposed and consistently developed by the author for more than 40 years at the V.A. Kotelnikov Institute of Radioengineering and Electronics (IRE) of RAS, the description and processing of signals and fields is carried out exclusively in the space of a fractional measure using scaling hypotheses, non-Gaussian stable distributions with heavy tails and, if possible, using the apparatus of fractional integroderivatives [1,5,7,16,17,34-84 , 99,100,107-109,113,123-133].

The evolution of the author's views and development at the moment in the IRE "fractal ideology" research is shown in **Fig. 19** and **Fig. 20**, which also provides information about the moment of their intensive deployment and open publications. All research is carried out by the author exclusively within the framework of a new fundamental interdisciplinary scientific direction, briefly designated as "Fractal radiophysics and fractal radio electronics: Design of fractal radio systems". In Fig. 19 introduced abbreviations: FNORS-fractal nonparametric detector of radar signals, FOS - fractal detector of signals.

Conventionally, three stages can be traced in these studies. At the *first stage,* the emphasis was placed on the experimental verification of the fractality of various natural and artificial formations, which made it possible to apply the concepts of fractional dimension and scale invariance to them, and to start developing methods for fractal filtering of objects in various intense noises. The *second stage* was entirely devoted to the improvement of the created original algorithms for fractal digital processing of signals and images, fractal methods of detection, recognition, enhancement of contrast, i.e. fractal

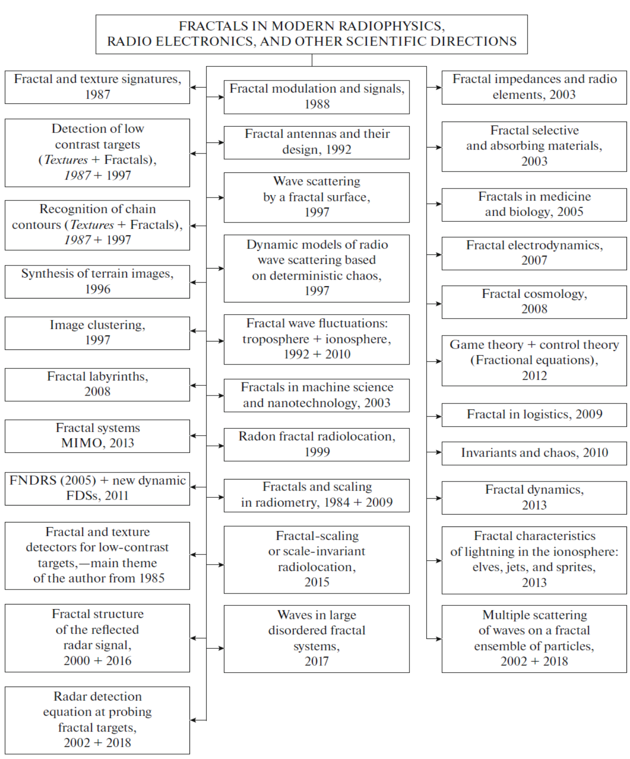


Fig. 19. *Sketch for the development of breakthrough technologies based on fractals, fractional operators and scaling effects for nonlinear physics and electronics.*

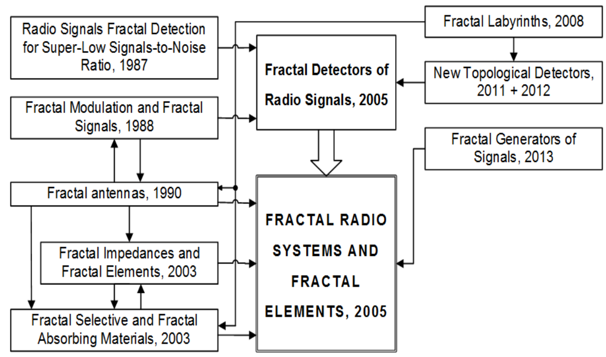


Fig. 20. *The author's concept of fractal radio systems, sensors, devices and radioelements.*

generalized filtering. The *third stage* is characterized by a transition to the design of a fractal element base and some fractal nodes, and in the future, fractal radio systems as a whole.

The analogy between the modern problems of radiophysics and radio electronics and the theory of phase transitions and critical phenomena is acquiring great importance. As is known, the modern renormalization group theory of phase transitions is based on an approach based on the scaling hypothesis, or scale invariance. On the basis of a deep study of this scientific direction, it was possible to form a similar approach for solving a large class of radiophysical and radio engineering problems.

Note that the presence of a fractional time derivative in the equations is interpreted as the presence of memory or, in the case of a stochastic process, non-Markovity.

3.2. Fractal measures and signatures

Fractals belong to sets with an extremely irregular branched or indented structure. The theory of fractals considers fractional measures instead of integer measures and is based on new quantitative indicators in the form of fractional dimensions D and corresponding fractal signatures. Fractal fractional dimensions D characterize not only the topology of objects, but also reflect the processes of evolution of dynamical systems and are related to their properties.

The classification of fractals developed by the author was approved in the USA in December 2005 and adopted by B. Mandelbrot [43,58,62]; it is shown in **Fig. 21**, where the properties of fractals are described, provided that D_0 is the topological dimension of the space in which a fractal with fractional dimension D is considered.

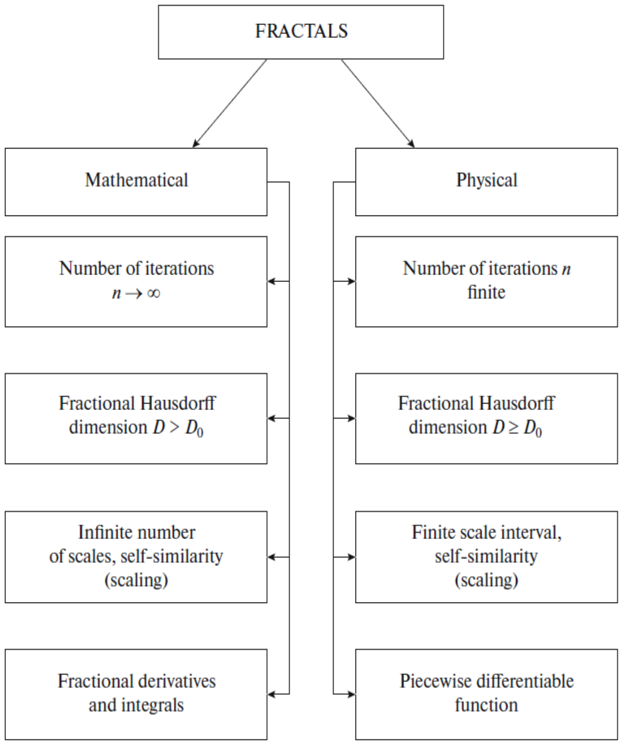


Fig. 21. *The author's classification of fractal sets and signatures, approved by B. Mandelbrot.*

Based on the data in Fig. 21, one can give a *mathematical definition* of a fractal.

A fractal is a functional mapping or set obtained by an infinite recursive process and having the following properties: 1) self-similarity or scale invariance (infinite scaling), that is, fractals on small scales look on average the same as on large ones; 2) fractional dimension (called Hausdorff dimension) strictly greater than topological dimension; 3) nondifferentiability and operation of fractional derivatives and integrals ".

The *physical* definition of a fractal is as follows:

"*Fractals* are geometrical objects (lines, surfaces, bodies) with a highly irregular structure and possessing the property of self-similarity on a limited scale."

The introduction by the author into the practice of measurements of the concepts of *fractal signatures* and *fractal cepstras* proved to be very fruitful [44,45,48,49,52-64,66-79,82,84,124-133]. The concept of "cepstrum" historically comes from the permutation of letters in the word "spectrum". The concept of "fractal cepstrum" is determined by the fact that when calculating the fractal dimension D of the received multidimensional signal, it is necessary to take the logarithm of the amplitudes averaged on different scales of the received time/space samples. Fractal signatures and fractal cepstras reflect the property of self-similarity of real signals and electromagnetic fields. Thus, in fractal processing methods it is always necessary to take into account the *scaling effects of real radio signals and electromagnetic fields*.

3.3. Texture and fractal processing of low-contrast images and ultra-weak signals in intense non-Gaussian interference and noise

In **Fig. 22** shows the complete structure of the author's research at the IRE of Russian Academy of Sciences of *texture* and *fractal* methods for processing low-contrast images and ultra-weak signals in intense non-Gaussian noise. The textural and fractal digital methods developed by the author make it possible to partially overcome the a priori uncertainty in radiophysical and radar problems using the *geometry* or *topology of the sample* - one-dimensional or multidimensional [17,57,58,62,82,84].

In this case, the topological features of the sample are of great importance, rather than the averaged realizations, which often have a different character. In order to focus on taking these features

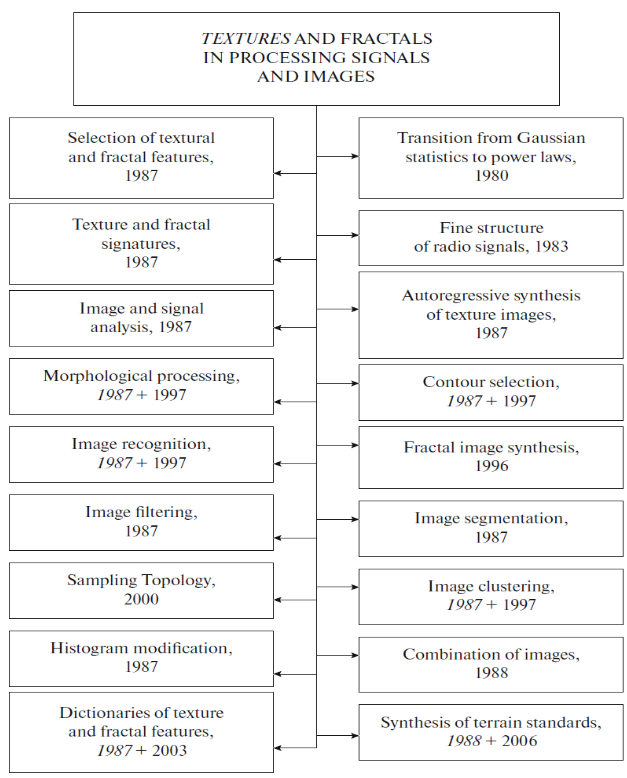


Fig. 22. *Classification of texture and fractal methods for processing low-contrast images and ultra-weak signals.*

into account, the term *dimensional sclerosis* of physical signals, fields and their fractal signatures was specially introduced [57,58,82]. When describing non-Markov processes, as is known [88], the term *asymptotic sclerosis* is widely used to reveal the physical meaning of fractional derivatives.

Texture is a matrix or fragment of spatial properties of image areas with homogeneous statistical characteristics [57,58]. Textural features (TP) are based on the statistical characteristics of the intensity levels of image elements and refer to probabilistic features, the random values of which are distributed over all classes of natural objects. The decision on whether a texture belongs to a particular class can be made only on the basis of specific values of the features of a given texture. In this case, it is customary to talk about t*he signature of the texture.*

Classic radar signatures include temporal, spectral and polarization features (features) of the reflected signal. The term "signature" describes the distribution of the general set of measurements for a given texture in scenes of the same type as this one [53]. In our experiments, we also optimized estimates of the effect of window size on the accuracy of determining texture features for images of various types of land cover. For a long time, the first works of the author in the field of research of joint radar (RI) and optical images (including synthetic aperture radars - SAR) of the earth covers using texture and fractal information, were actually the only ones in the USSR and Russia, and today they are also not lost relevance.

On the basis of the results obtained by the author, the following unconventional and rather effective methods of signal detection at small signal-to-background ratios were first proposed and implemented: q_0^2*-the dispersion method, the detection method using linearly modeled standards, and the method with direct use of the ensemble of texture features* [44].

We especially note that the developed fractal (topological) methods constitute an independent area of research and are not directly related either to classical probability distributions of mathematical statistics, or to the classical theory of outliers, or to questions of statistical topography of random processes and fields.

If, in syntactic or structural recognition, the structure of objects, its hierarchy and connections between them are investigated, then in *fractal recognition,* the topology of the object and the background, displayed in one-dimensional and multidimensional received radar signals, is investigated. With the fractal approach, it is necessary to search, implement and use the rules that obey the fractional (complex) topology of the images under consideration. Then the procedure of *fractal recognition* is a comparison with the dictionary of fractal features [57,58,62,77,82].

In this case, we will highlight the "*fractal primitives*" - the elements of the "*fractal language*". The question inevitably arises about the composition of fractal primitives - *fractal symbols*, which are the smallest elements of a fractal language. We will call the set of fractal symbols used "*fractal alphabet*" or "*fractal dictionary*", denoted by the symbol Φ. On the basis of the latter, one can compose "*fractal strings*" - finite sequences of symbols included in the alphabet. The string can be any length. All possible strings of the fractal alphabet form a universal set of strings or a closure Φ. If we introduce a set of empty lines, then a finite or countable infinite subset of the closure of the fractal alphabet Φ is a more precise definition of the concept of "*fractal language*". Separate fractal lines, composed of its fractal symbols, we will call "*fractal words*".

Further, performing some logical operations on a fractal language, you can create a new language. The rules for creating, transforming and interacting fractal words will be determined by the "*fractal grammar*". For its construction, it is possible to use the ideas of formal grammar developed in mathematical linguistics.

3.4. Formal fractal grammars

Formal grammar methods are characterized by two features [15]. First, they describe only a set of possible results and do not give direct instructions on how to get the result for a particular task. Secondly, in them all statements are formulated exclusively in terms of a small number of well-defined and elementary symbols and operations. Therefore, formal grammars are simple from the point of view of their logical construction.

A formal grammar can be defined by a generative grammar - the system

$$G = \langle C_t, C_n, P, A \rangle, \tag{155}$$

consisting of four parts: terminal (main) dictionary C_t, nonterminal (auxiliary) dictionary C_n, set of substitution rules P, initial symbol or initial axiom $A(A \in C_n)$.

Terminal (main) dictionary C_t is a set of non-derivative terminal elements or features from which the chains generated by the grammar are built. The choice of non-derivative elements refers to the problem of determining informative and stable features for recognition. Nonterminal (auxiliary) vocabulary C_n - a set of symbols that denote classes of source elements or chains of source elements, as well as some special nonterminal or auxiliary elements. The initial symbol A is a distinguished nonterminal symbol denoting a collection or class of all those linguistic objects for which this grammar is intended to describe (for example, in a grammar that generates sentences, the initial symbol is a symbol meaning a sentence, etc.). The set of substitution rules P is a finite set of rules of the form $\varphi \rightarrow \psi$, where φ and are words in the dictionary (alphabet) $C_n \cup C_t$ and "→" is a symbol that does not belong $C_n \cup C_t$. Generative grammar is not an algorithm because substitution rules are a collection of solutions, not a sequence of prescriptions.

Formal grammar is characterized by the following ratios:

$$C = C_n \cup C_t,\ C_n \cap C_t = 0, \tag{156}$$

where C is a dictionary.

The process of creating a language begins with Axiom A, to which the substitution rules are applied one by one. Conjunction, disjunction, negation are used as operations on statements.

Let's consider an example of the formation of some *fractal primitives.* In practice, to obtain a fractal grammar, it is necessary to derive it from a given ensemble of learning objects. This procedure is similar to the learning problem in various recognition methods. Algorithms for fractal pattern recognition are based on the use of the "t*arget topology - its fractal dimension*" paradigm. The a priori space of deterministic or probabilistic features is determined using a dynamic test.

The selection and preparation of test material for experimental verification of fractal recognition methods and verification of the principles of constructing algorithms significantly affect the reliability of the research results. In the vast majority of problems arising in practice, the classical methods of the theory of statistical decisions are of little use for recognizing radar targets. This is due to the fact that there are: severe restrictions on the analysis time, bandwidth of information transmission channels; high level of a priori uncertainty; the impact of various kinds of interference, a wide variety of characteristics of targets combined into one class, unknown target orientation; the simultaneous presence of several targets, differing in orientation and size.

The formalization of the problem under consideration classically presupposes the following stages: (**i**) - the initial a priori classification of goals or their classes, i.e. compilation of an alphabet of classes of goals; (**ii**) - determination of the required list of features that characterize the goals (in this case, we are talking only about fractal features); (**iii**) - development of a reference dictionary of fractal attributes of goals or classes of goals; (iv) - description of the alphabet of target classes in the language of the ensemble of fractal features of the reference dictionary or their combinations; (v) - partition of the space of fractal features into regions corresponding to the original classes of the alphabet; (**vi**) - the choice of a metric (decision rule) or recognition algorithms that ensure the assignment of a recognized target to one or another class of targets. When developing the first reference dictionary of fractal features, the latter were selected: 1) - the value of the fractal (fractional) dimension D; 2) - the type of fractal signatures or fractal cepstras; 3) - the type of the spatial spectrum and the values of spatial frequencies that characterize the texture of the images [57,58,62,77,82].

It is proved that fractal cepstrum is, on the one hand, a convenient topological invariant - it does not require preliminary orientation/scaling, and on the other hand, it is insensitive to image contrast. Thus, the positions of the characteristic points on fractal cepstras make it possible to determine the class of the target (according to some rule), its size, as well as the number of targets. The relative change in the position of the characteristic points makes it possible to solve the problem of detecting a deterministic target even with very low contrast.

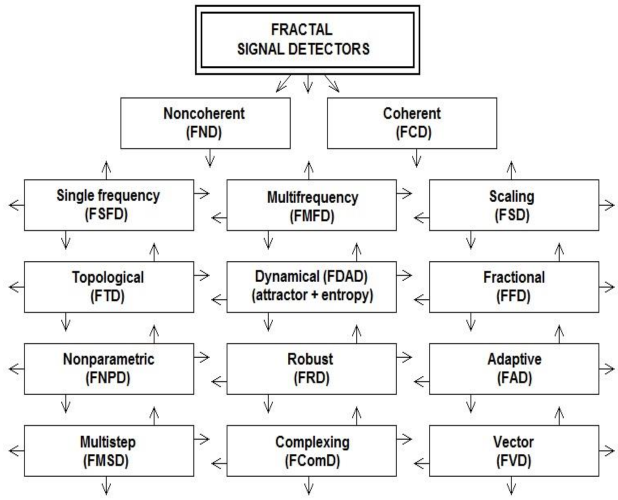


Fig. 23. *The main types of the proposed fractal-scaling or topological signal detectors.*

3.5. Fractal radar signal detectors

Creation of the first reference dictionary of fractal features of target classes and constant improvement of algorithmic support were the main stages in the development and prototyping of the first fractal nonparametric radar signal detector (FNORS) in the form of a special processor [16,17,47,57,58,62,77,82-84]. The main types of the proposed topological signal detectors are shown in **Fig. 23**.

3.6. Adaptation of fractal detectors

The creation of adaptive methods as applied to fractal information processing is of great interest. As is known [87], an adaptive problem is characterized by a change in the parameters and / or structure of the system in accordance with external conditions. Some ways of obtaining theoretical and technical solutions to the problem of synthesis of adaptive fractal detectors are shown below [42,46,57,58].

Working with signal sampling against the background of noise and noise in the space of fractional measure, we inevitably come to algorithms (criteria) of adaptive fractal filtering. The adaptation of such nonlinear fractal filtering under conditions of a priori uncertainty is provided, in particular, by the current estimate of the Hurst exponent N. As noted above, the Hurst exponent, depending on its value relative to the value $H = ½$, characterizes either persistence (½) or antipersistence ($0 < H < ½$) sampling.

In the first case, when $1/2 < H < 1$, we observe a process that maintains the tendency of an increase or decrease in instantaneous amplitudes in the sample, i.e. process with memory. In the second case, when $0 < H < 1/2$, an increase in the amplitudes of the signal envelope in the “past” means a decrease in the “future”, and vice versa, i.e. a process that is more subject to change, often referred to as “return to the mean”.

Fixing the value of H in terms of [87] is a *counter hypothesis* that improves the quality of adaptation to real conditions. In the general case, the device is a tracking system that adapts the values of the Hurst exponent N to the interference situation or, conversely, to the useful signal. An example of an adaptive procedure is the automatic control of the receiver gain depending on the current estimate $H = f(t)$. In another adaptive procedure, the detection threshold P is automatically adjusted according to the values $H = f(t)$. This ensures the stabilization of the false alarm probability.

3.7. Fractal scaling or scale invariant radar

The detection of low-contrast objects against the background of natural intense interference inevitably requires the calculation of a fundamentally new characteristic, which differs from the functionals associated with interference and signal energy, and is determined only by the topology and dimension of the received signal. The introduction of the concepts of "deterministic chaos", "texture", "fractal" and "fractal dimension D" into the scientific use of radar [17,36,57,58,62,77,82-84] allowed us for the first time in the world to propose, and then and apply new dimensional and topological (not energy!) features or invariants (**Fig. 24**), which are combined under the generalized concept of "sample topology" ~ "fractal signature".

Fractal-scaling or scale-invariant radar [17,47,52,54,55,61,66-68,82-84,125,127,130,131] is based on three *postulates:* 1 – intelligent signal/ image processing based on fractional measure theory and scaling effects, for calculating the field of fractal dimensions; 2 – the sample of the received signal in noise belongs to the class of stable non-Gaussian probability distributions D of the signal; 3 – maximum topology with minimum energy of the input random signal. These postulates open up fundamentally new possibilities for ensuring stable operation at short q_0^2 or increased ranges of radars. Algorithms for detecting extended objects and targets on optical and radar images using texture

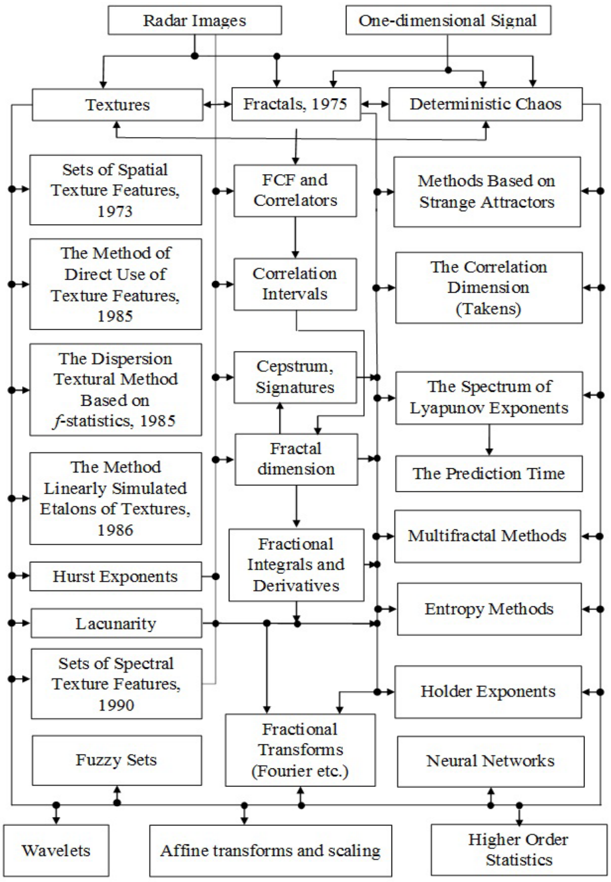


Fig. 24. *New topological features and methods for detecting low-contrast objects against the background of interference (TP - texture features, PFC - frequency coherence function).*

processing were created by us back in the 80s of the XX century (see the left column in Fig. 24).

3.8. Main results

As a result of joint long-term field experiments with leading industry research institutes and design bureaus of the USSR and Russia, a statistical analysis of large arrays of new data on the spatial and temporal characteristics of earth cover scattering in the IIM and CMB ranges was carried out, taking into account their seasonal and angular variations in various meteorological conditions in order to assess boundaries of radar contrasts, distribution laws of specific RCS, spectrum width, time and correlation interval of intensity fluctuations of reflected simple and complex phase-shift keyed signals in the IIM range and the structure of reflected pulse signals, which makes it possible to take into account terrain features when designing various imaging systems.

A theory of scattering of millimeter radio waves by chaotic covers has been developed, using the first introduced functionals of backscattering stochastic fields and frequency coherence functions (PFCs) taking into account the antenna directional pattern and correlation of the slopes of irregularities. The results of this theory make it possible to determine the coherence bands of space-time radio channels with variable parameters for the optimal choice of the width of the sounding signal spectrum, frequency spacing in multifrequency systems and the base value of complex sounding signals, characteristics of reflected signals, generalized uncertainty functions, potential accuracy of estimates of the flight altitude, and the characteristic dimensions of the irregularities. Theoretical and experimental results were used in the synthesis of reference digital radar maps of the area.

For the first time, a new class of informative features has been proposed, based on the fine structure of reflected millimeter-wave radar signals, and allowing to improve the identification of earth covers.

For the first time, complete ensembles of texture and spatial correlation-spectral features of optical and radar images of real earth covers were investigated, followed by the selection of clusters and the determination of the most informative features for certain classes of textures. It is proved that the area of existence of texture features of radar images is completely determined by the corresponding areas of features of optical images. The experiments carried out have demonstrated the efficiency and generality of the proposed approach in the problems of classification of the earth's covers when integrating images at optical and millimeter waves. Image integration increases the efficiency of detection, recognition and classification based on an extended vector of informative and stable features. Image processing results in detailed digital radar maps of the area. Such maps make it possible to present radar information in a form convenient for further use in radio navigation of aircraft and recognition of various types of ground objects. [Note that these studies had no analogues, both in the USSR and in Russia, and have not lost their relevance at the present time].

For the first time, a number of texture methods have been developed (Fig. 22, Fig. 24) for detecting various objects and their contours on real optical and radar images of the earth's surface at low signal /background ratios. A relationship is established between the size of the object and the analyzed fragment of optical and radar images of a wide class of earth covers in the case of optimal detection. The possibility of stochastic autoregressive synthesis of optical and radar images of earth covers with the operation of converting brightness histograms has been theoretically substantiated and experimentally confirmed. The optimal sizes of the brightness space and the order of autoregressive series involved in forecasting for adequate synthesis of images have been determined. With an increase in the order of correlation, the areas for determining the textural features of synthesized images are narrowed. When comparing sections of the original optical or radar image with the synthesized standard, it is shown that the final two-dimensional binary field of cross-correlation coefficients directly fixes the location of the object in the original image. This allows you to form a motion map and the dynamics of the detected object. It was established using various matching algorithms (classical correlation, pair function method, absolute difference method) that the physical reliability of stochastic autoregressive synthesis reaches 90%.

A system approach to the formation of an information-axiomatic model of radar maps of heterogeneous terrain has been developed and implemented on the basis of the above radiophysical studies. A generalized radiophysical model for the formation of radar maps of heterogeneous terrain has been created, which includes both methods of stochastic autoregressive synthesis of images, and information about the field of specific EPR of the earth's covers. The characteristic number of gradations of the specific EPR of the earth's surface has been established. Based on the analysis of the system architecture to obtain a standard, an algorithm for the synthesis in the radio range of contour and grayscale radar maps of inhomogeneous terrain is implemented. It is shown that the destruction of the correlation maximum occurs for a contour radar map of the area at a wavelength of 8.6 mm at an angle of mutual rotation of 5°… 7°, and for a half-tone radar map - at an angle within 14°… 17°. Then, fractal parameters were first introduced into the generalized radiophysical model of the formation of radar maps of heterogeneous terrain, which increased the information content of the synthesis.

The presence of a strange attractor is predicted to control radar scatter from vegetation. Subsequently, the effect was discovered experimentally at a wavelength of 2.2 mm (2002). The results obtained confirmed the theoretical concepts of the existence of a chaos regime in a dynamical system that describes the nature of the scattering of electromagnetic waves by vegetation [81].

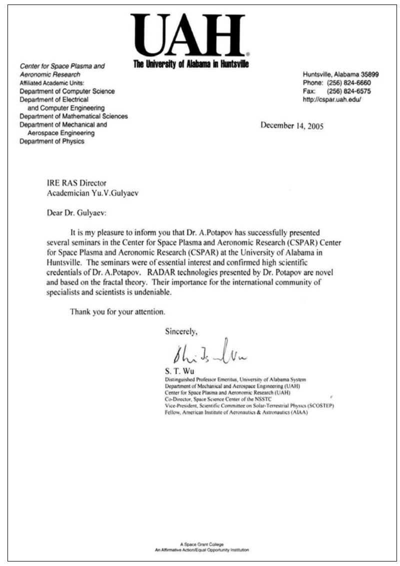
Reconstruction of the attractor made it possible to determine its fractal dimension *D*, the maximum Lyapunov exponent, the dimension of the embedding, and the prediction interval (time). The experimental characteristics of the strange attractor formed the basis for a fundamentally new non-Gaussian model of radar scattering of the IMV by vegetation covers based on the theory of dynamical systems and stable distributions. It is shown that the interval (time) for predicting the intensity of the reflected radar signal exceeds the classical correlation time by about an order of magnitude. This made it possible to introduce into the theory of radar a new essential characteristic, namely, the prediction interval (time), which expands the methods and circuitry of radars.

A reliable physical substantiation of the practical application of fractal methods (Fig. 19 - Fig. 24) in modern fields of radiophysics, radio electronics and information control systems is given.

In the mid 80s. XX century. A working model of a coherent compact digital solid-state radar (TsTR) based on parametrons with a probing wavelength of 8.6 mm with a complex signal base> 106 and with processing of an input subnoise signal at a carrier frequency was created jointly with the Almaz Central Design Bureau. With optimal processing, the energy potential of the CTR increased by 50 dB. Then, a DTC was created at two sounding frequencies in the MMV and CMB bands with a fractal slot antenna (the first in the USSR); for the synthesis of images, the Radon transform is used. In 1997, the first developed methods of fractal modulation and fractal signals, including the first introduced by the author H-signals.

Together with representatives of the Central Design Bureau "Almaz" A.A. Potapov was one of the co-leaders of the international project No. 0847.2 through the ISTC (2000-2005) to create a multifunctional automated radio measuring system with a complex signal at centimeter and millimeter waves, using fundamentally new patented circuitry and digital information processing technologies based on fractal and Radon algorithms in real time. Giving lectures on fractal and texture technologies developed by him at IRE RAS and reports on the ISTC project in 2000 and 2005 in the USA (Washington, New York, Huntsville, Atlanta, Franklin), in China (2011 to the present) time) and at numerous International conferences (England, USA, Canada, Holland, Austria, Germany, France, Spain, Italy, Hungary, Greece, Turkey, Scotland, Switzerland, Sweden, Mexico, China, Serbia, Montenegro, Bulgaria, Kazakhstan, Belarus, Ukraine) brought him wide fame in the circles of the international scientific community.

In December 2005, American specialists (from the University of Alabama and the Center for Space Plasma and Aerial Research of the United States) in an official letter addressed to the Director of the IRE RAS Academician Yu.V. Gulyaev, it was noted that "... *The seminars were extremely interesting and confirmed the high scientific qualifications of Dr. A. Potapov. Radar technologies presented by Dr. A. Potapov are based on the theory of fractals and are new. The importance of these studies for the international community of specialists and scientists is undeniable*" – (**Fig. 25**).


UAH
The University of Alabama in Huntsville

Center for Space Plasma and Aeronomic Research
Affiliated Academic Units:
Department of Computer Science
Department of Electrical and Computer Engineering
Department of Mathematical Sciences
Department of Mechanical and Aerospace Engineering
Department of Physics

Huntsville, Alabama 35899
Phone: (256) 824-6660
Fax: (256) 824-6575
http://cspar.uah.edu/

December 14, 2005

IRE RAS Director
Academician Yu.V.Gulyaev

Dear Dr. Gulyaev:

It is my pleasure to inform you that Dr. A.Potapov has successfully presented several seminars in the Center for Space Plasma and Aeronomic Research (CSPAR) Center for Space Plasma and Aeronomic Research (CSPAR) at the University of Alabama in Huntsville. The seminars were of essential interest and confirmed high scientific credentials of Dr. A.Potapov. RADAR technologies presented by Dr. Potapov are novel and based on the fractal theory. Their importance for the international community of specialists and scientists is undeniable.

Thank you for your attention.

Sincerely,

S. T. Wu
Distinguished Professor Emeritus, University of Alabama System
Department of Mechanical and Aerospace Engineering (UAH)
Center for Space Plasma and Aeronomic Research (UAH)
Co-Director, Space Science Center of the NSSTC
Vice-President, Scientific Committee on Solar-Terrestrial Physics (SCOSTEP)
Fellow, American Institute of Aeronautics & Astronautics (AIAA)

A Space Grant College
An Affirmative Action/Equal Opportunity Institution

Fig. 25. *Letter from the US Space Plasma Center.*

At the same time, a significant and long-term scientific meeting of A.A. Potapov with the founder of fractal geometry B. Mandelbrot at his home in the USA, when he accepted and approved the definition of fractals introduced by A.A. Potapov in his books and articles, and his work (**Fig. 26**).

Fig. 26. *B.B. Mandelbrot and A.A. Potapov. New York, USA, 2005*

For the first time, the effectiveness and prospects of applying the theory of fractional measures and scaling relations (for textures and fractals) in the case of detection and recognition (generalized filtering) of one-dimensional and multidimensional radar signals from low-contrast targets against the background of intense non-Gaussian interference of various kinds was discovered and proved. Thus, this is a *fundamentally new* radioengineering.

It has been proved that when collecting, transforming and storing information in modern complex systems for monitoring remote and mobile objects in conditions of intense interference, the newest methods of processing information flows and multidimensional signals, proposed by the author, are of great importance. Typically, the features of such complex systems are manifested at different spatio-temporal scales. The most adequate assessments of the states of the system under study and the dynamics of changes in the state of its subsystems are realized using the theory of fractals and processing multidimensional signals in a space of fractional dimension with the indispensable consideration of the effects of scaling, which was first proposed and developed by the author at the IRE RAS.

A new method of measuring fractal dimension and corresponding fractal signatures of signals, images and wave fields is proposed and substantiated. This method, as well as its effectiveness, has been confirmed in practice by numerous examples of appropriate digital processing of optical and radar natural and synthesized images, including those with low-contrast objects. Texture and fractal digital methods (Fig. 19 and Fig. 22) make it possible to partially overcome the a priori uncertainty in radar problems using the *geometry* or *topology of the sample* - one-dimensional or multidimensional. In this case, the topological features of the sample are of great importance, rather than the averaged realizations, which often have a different character.

Methods of fractal classification, clustering and recognition of many types of natural and artificial objects have been studied for the first time on large arrays of experimental data in the form of optical and radar images of real earth covers with surface and subsurface objects. The number of areas around which the fractal dimension values are grouped depends on the parameters of the algorithm and the measurement method. For example, with a small size of the measuring window, we have a large number of groups; increasing the size, we get a fixed number of groups or clusters; and finally, with a very large window size, 2 - 3 groups remain (fractal objects - non-fractal objects - exclusion objects).

The study of the type or topology of sampling a one-dimensional (multidimensional) signal for tasks, for example, artificial intelligence, for the first time made it possible to create dictionaries of fractal features based on fractal primitives, which are elements of a fractal language with fractal grammar. The data obtained were incorporated into the synthesis of reference and current radar maps of heterogeneous terrain, as well as into non-energy radar detectors.

The results (UAV, SAR, medicine, etc.) show that fractal processing methods increase the quality and detail of objects and targets in passive and active modes by several times. These methods can be successfully applied to process information from space, aviation complexes, low-signature high-altitude pseudo-satellites (HAPS) or the detection of HAPS and UAV clusters, synthesized clusters of space antennas and space debris.

The fractal characteristics of elves, jets and sprites - the most interesting types of recently discovered high-altitude discharges in the ionosphere - are investigated.

Algorithms for detecting a moving distant object of unknown shape (fractal or non-fractal) on a low-contrast image formed in optoelectronic systems have been synthesized with co-authors. Experimental results on images obtained in field conditions confirm the effectiveness of the proposed processing methods.

For the first time, the fundamental possibility of synthesizing new fractal functions and fractal functionals on the basis of the theory of fuzzy sets has been proved. Formalized the construction of new classes of fractal and multifractal subsets on fuzzy sets. Any classic non-differentiable functions can be used as trial functions.

It is shown for the first time that the physical content of the theory of diffraction, containing multiscale surfaces, becomes clearer with the fractal approach and the separation of the fractal dimension D or the fractal signature as a parameter. Taking into account fractality significantly brings the theoretical and experimental characteristics of the scattering indicatrices of the earth's covers closer together, which is important for the problems of radar and remote sensing. For the first time in the world, an extensive catalog of characteristic types of more than 70 fractal surfaces based on Weierstrass functions, as well as more than 70 three-dimensional scattering indicatrices and their cross sections calculated for wavelengths $\lambda = 2.2$ mm, $\lambda = 8.6$ mm and $\lambda = 3$ cm at different values of fractal dimension D and varying scattering geometry.

Analogs of Maxwell's equations with fractional Caputo derivatives are obtained. Gauge invariance is considered and a diffusion-wave equation for scalar and vector potentials is derived. A particular solution of the diffusion-wave equation is found and analyzed [9,10].

A rigorous electrodynamic calculation of numerous types of fractal antennas was carried out, the design principles of which underlie fractal frequency-selective surfaces and volumes (fractal “sandwiches”).

A family of broadband miniature fractal antennas is synthesized based on the topology of fractal labyrinths. The author proposed to synthesize large stochastic robust antenna arrays using the properties of fractal labyrinths. Combining several fractal labyrinth clusters with different fractal dimensions makes it possible to create adaptive broadband fractal antennas. For the first time, a model of a “fractal” capacitor as a fractal impedance is proposed and implemented.

Created, substantiated and applied fractal-scaling methods for radar problems and the formation of the foundations of a fractal element base, fractal sensors and fractal radio systems. A physical approach to modeling a fractal capacitor and fractal impedances has been developed. Promising elements of fractal radio electronics are functional elements, the fractal impedances of which are realized on the basis of the fractal geometry of conductors on the surface (fractal nanostructures) and in space (fractal antennas), fractal geometry of the surface microrelief of materials, etc. The developed approaches can be extended to a wide class of electrodynamic problems in the study of fractal magnonic crystals, fractal resonators, fractal screens and barriers, as well as other fractal frequency-selective surfaces and volumes.

A new type and new method of modern radar, namely, fractal-scaling or scale-invariant radar, has been discovered, proposed and substantiated. The efficiency of functionals, which are determined by the topology, fractional dimension and texture of the received multidimensional signal, for the synthesis of fundamentally new non-energy detectors of low-contrast objects against the background of noise has been proved (Fig. 19, Fig. 20, Fig. 23). An increase in the sensitivity of the radio system (which is equivalent to an increase in the range) was confirmed when using fractal and texture features in topological detectors. This entails fundamental changes in the very structure of theoretical radar, as well as in its mathematical apparatus.

Fractal radar is capable of adequately describing and explaining a much broader class of radar phenomena. The scientific direction created for the first time in Russia and in the world is based on the concept of fractal radio systems and fractal radioelements, sampling topology and the global fractal-scaling method, proposed and created by the author at the V.A. Kotelnikov IRE of RAS. The research carried out in the field of theoretical radar makes it possible to effectively solve the problems of signal detection in conditions of intense interference and create new fractal multifrequency MIMO systems

The postulates of fractal scaling radar have been developed: 1 – intelligent signal/image processing based on the theory of fractional measures and scaling effects for calculating the fractal dimension field; 2 – the sample of the received signal in noise belongs to the class of stable non-Gaussian probability distributions D of the signal; 3 – maximum topology with minimum energy of the input random signal (ie, maximum "deviation" from the energy of the received signal).

These postulates open up new possibilities for ensuring stable operation at low signal / (noise + interference) ratios or increasing the range of radars.

Theoretical issues of fractal non-inertial relativistic radar and quantum cosmology in curved space-time of negative fractal dimension have been significantly developed together with colleagues from Russia (Moscow, VNIIOFI) and Israel (Haifa, Technion). *Example:* on the basis of the Schrödinger equation with the operator of fractional calculus in spatial coordinates, the Feynman integral over trajectories is calculated for the generalized Lagrangian with the operator of fractional differentiation in time. Note that at present in the United States this fundamental scientific direction has received the bright name "*Fractal Cosmology*" [34,35,82,99,108,123].

4. CONCLUSION

The results of experimental and theoretical research obtained by the author and his students were introduced by the leading industry research institutes and design bureaus of the USSR and Russia and were used in the design of radio systems for various purposes, in the interpretation of data from remote radiophysical studies of the environment and in other applied problems in which optical and radar systems serve as information materials. images of the earth's surface.

On the basis of many years of research, new theoretical directions in the theory of statistical decisions, statistical radio engineering and statistical radiophysics have been formulated and developed, for example, "Fractal analysis and its application in the theory of statistical decisions and statistical radio engineering", "Statistical theory of fractal radar", "Statistical fractal radio engineering", "Theoretical foundations of fractal radar", etc.

The above results formed the basis of the fractal paradigm and a single global idea of fractal natural science [73, 82].

The performed studies are a priority in the world and serve as a basis for further development and substantiation of the practical application of the fractal-scaling and texture methods created by the author in modern radiophysics and radar, as well as in the creation of fundamentally new and more accurate topological fractal-texture methods for detecting and measuring parameters of radio signals in spatio-temporal radar channel of propagation of electromagnetic waves with scattering.

The radio engineering "fractal geometry" of the transceiver or any information radio engineering, optoelectronic and acoustoelectronic systems, along with fractal modulation/demodulation and cryptographic resistance methods (fractal ultra-wideband signals, fractal information compression [57,58]) are extremely promising measures for solving urgent problems of traditional radio electronics, which since its inception is completely based on an integer measure.

These issues are relevant in solving the problems of constructing and optimizing the characteristics of modern and promising radio-physical intelligent sensing systems for detecting and recognizing various objects in difficult conditions using topological fractal and texture methods based on the previously proposed general principles of fractal-scaling or scale-invariant radar [57, 58.82.84.125-132.137.138]. It should be noted that *fractal radars* are, in fact, a necessary intermediate stage on the path of transition to *cognitive radar* and *quantum radar*. Note that our recent results with our Chinese colleagues on the effects of microscale optoelectronics and photonics [107,139-145] will help open the way for controlling light scattering using magnetoelectric couplings and previously unknown wave phenomena in order to design new devices for processing multidimensional signals in such intelligent systems.

Based on the author's monographs, courses of lectures on fractals in radiophysics and radio electronics have been delivered at various universities in Russia and neighboring countries, as well as in China. The author's priority in the above scientific areas is secured in the world by more than 1150 scientific works, including 45 domestic and foreign monographs and individual chapters in them in Russian and English, and 2 patents (see, for example, [82,83,139]).

REFERENCES

1. Akinshin RN, Potapov AA, Rumyantsev VL and others. *Physical foundations of the device of missile and artillery weapons. Algorithms and devices for the functioning of airborne radio-technical means of air reconnaissance of artillery.* Penza: Branch of VA MTO, Penz. art. Ing. Institute, 2018, 400 p.
2. Alexandrov PS. *An introduction to set theory and general topology.* Moscow, Nauka Publ., 1977, 368 p.
3. Alexandrov PS, Pasynkov VA. *An introduction to dimension theory*. Moscow, Nauka Publ., 1973, 576 p.
4. Babenko YI. *Fractional Differentiation Method in Applied Problems of Heat and Mass Transfer Theory.* SPb, NPO "Professional" Publ., 2009, 584 p.
5. Bagmanov VH, Potapov AA, Sultanov AH, Wei Zhang. Fractal filters for signal detection in remote sensing data processing. *Radioengineering and electronics,* 2018, 63(10):1062–1068.
6. Bardou F, Buscho J-F, Aspe A, Cohen-Tannoudji K. *Levy statistics and laser cooling. How rare events stop atoms.* Moscow, Fizmatlit Publ., 2006, 216 p.
7. Bekmachev YES, Potapov AA, Ushakov PA. Designing fractal proportional-integral-differential fractional order controllers. *Uspekhi sovrem. radioelectr.*, 2011, 5:13-20.
8. Bogachev VI. *Foundations of measure theory.* Moscow-Izhevsk, Research Center Regular and Chaotic Dynamics Publ., 2006, vol. -544 p.; vol. 2-576 p.
9. Bogolyubov AN, Potapov AA, Rekhviashvili SS. Interpretation of the solution of the diffusion-wave equation using fractional integro-differentiation. *Vestn. Moscow univ. Ser. 3. Physics. Astronomy,* 2010, 3:54-55 (in Russ.).
10. Bogolyubov AN, Potapov AA, Rekhviashvili SS. Method of introducing fractional integro-differentiation in classical electrodynamics. *Vestn. Moscow univ. Ser. 3. Physics. Astronomy,* 2009, 4:9-12.
11. Baer R. *Theory of discontinuous functions.* Moscow-Leningrad, ONTI Pudl., 1932, 136 p.
12. Voroshilov AA, Kilbas AA. Cauchy problem for diffusion-wave equation with partial derivatives of Caputo. *Differentsialnye uravneniya*, 2006, 42(5):595-609 (in Russ.).
13. Gelfand IM, Shilov GE. *Obobshchennye funktsii i deystviya nad nimi* [Generalized functions and actions on them]. Moscow, Fizmatlit Publ., 1958, 440 p.
14. Gnedenko BV, Kolmogorov AN. *Predel'nye raspredeleniya dlya summ nezavisimykh sluchainykh velichin* [Limit distributions for sums of independent random variables]. Moscow-Leningrad, GITTL Publ., 1949, 264 p.

15. Gorelik AP, Skripkin VA. *Metody raspoznavaniya* [Recognition methods]. Moscow, Vysshaya shkola Publ., 1989, 232 p.
16. Gulyaev YuV, Nikitov SA, Potapov AA, German VA. Scaling and fractional dimension ideas in a fractal radio signal detector. *Radio engineering and electronics,* 2006, 51(8):968-975.
17. Gulyaev YuV, Potapov AA. Application of the theory of fractals, fractional operators, textures, scaling effects and methods of nonlinear dynamics in the synthesis of new information technologies for problems of radio electronics (in particular, radar). *Radio engineering and electronics*, 2019, 64(9):839-854.
18. Gurevich V, Volman G. *Dimension theory.* Ed. and with a foreword. PS. Alexandrova. Moscow, IL Publ., 1948, 232 p.
19. Erofeev VI, Potapov AA. International scientific colloquium "Mechanics of generalized continua: one hundred years after Cosserat". *Nelineinyi mir,* 2009, 7(8):652-654 (in Russ.).
20. Zolotarev VM. *One-dimensional stable distributions*. Moscow, Nauka Publ., 1983, 304 p.
21. Kantor G. *Trudy on set theory*. Eds. AN Kolmogorov and AP Yushkevich. Moscow, Nauka Publ, 1985, 432 p.
22. Kobelev VL, Romanov EP, Kobelev YL, Kobelev LYa. Non-Debye relaxation and diffusion in fractal space. *DAN,* 1998, 361(6):755-758.
23. Kobelev YL, Kobelev L Ya, Romanov EP. Autowave processes in nonlinear fractal diffusion. *Dokl. RAS*, 1999, 369(3):332-333.
24. Kolmogorov AN, Fomin SV. *Elements of the theory of functions and functional analysis.* Moscow, Vysshaya shkola Publ., 1989, 624 p.
25. Letnikov AV. The theory of differentiation with an arbitrary pointer. *Mat. collection*, 1868, no. 3:1-68.
26. Medvedev FA. *Essays on the history of the theory of functions of a real variable.* Moscow, Nauka Publ., 1975, 248 p.
27. Medvedev FA. *The development of set theory in the 19th century.* Moscow, Nauka Publ., 1965, 232 p.
28. Milnor J. *Holomorphic Dynamics.* Per. from English. Izhevsk, Research Center "Regular and Chaotic Dynamics" Publ., 2000, 320 p.
29. Morozov AD, Dragunov TN. *Visualization and analysis of invariant sets of dynamical systems.* Izhevsk, Institute of Computer Research Publ., 2003, 304 p.
30. Nakhushev AM. *Fractional calculus and its application*. Moscow, Fizmatlit Publ., 2003, 272 p.
31. Nakhushev AM. *Elements of fractional calculus and their application*. Nalchik, Ed. KBNTs RAN Publ., 2000, 299 p.
32. Novozhenova OG. *Biography and scientific works of Alexei Nikiforovich Gerasimov. On linear operators, elastic viscosity, eleutherosis and fractional derivatives*. Moscow, Perot Publ., 2018, 234 p.
33. Stepson BA, Fedorchuk VV, Filippov VV. *Dimension theory. Results of Science and Technology. Series: Algebra. Topology. Geometry.* Moscow, VINITI Publ., 1979, 17:229-306.
34. Podosenov SA, Potapov AA, Sokolov AA. *Pulse electrodynamics of broadband radio systems and fields of coupled structures.* Ed. AA Potapov. Moscow, Radiotekhnika Publ., 2003, 720 p.
35. Podosenov SA, Potapov AA, Foukzon J, Menkova EP. *Nonholonomic, fractal and coupled structures in relativistic continuous media, electrodynamics, quantum mechanics and cosmology.* Ed. AA. Potapov. Moscow, LENAND, URSS Publ., 2015, in 3 vols., 1128 p.
36. Potapov AA. Waves in disordered large fractal systems: radar, nanosystems, clusters of unmanned aerial vehicles and small spacecraft. *Radio Engineering and Electronics*, 2018, 63(9):915-934.
37. Potapov AA. Diffractals at a frequency of 36 GHz observed during radar scattering of an electromagnetic wave by a fractal surface, and wave catastrophes in fractal randomly inhomogeneous media. *Proc. XIII Int. conf. "Zababakhin Scientific Readings", dedicated. 100th anniversary of the birth of EI Zababakhin* (Snezhinsk, March 20-24, 2017). Snezhinsk, RFNC-VNIITF Publ., 2017, p. 137-138.
38. Potapov AA. Fractional and integer topological dimensions as the main components in the topology of multidimensional signal sampling and their processing. *Proc. Int. conf. "Differential equations and topology", dedicated. 100th anniversary of the birth. acad. LS Pontryagin* (Moscow, June 17-22, 2008). Moscow, Steklov Mat. inst. of RAS and Lomonosov Moscow State University (MAKS Press), 2008, pp. 478-479.
39. Potapov AA. On the theory of functionals of backscattering stochastic fields. *Radio Engineering and Electronics,* 2007, 52(3):261-310.
40. Potapov AA. A short historical essay on the origin and formation of the theory of fractional integrodifferentiation. *Nelineinyi mir*, 2003, 1(1-2):69-81.
41. Potapov AA. Multiple scattering of waves by a fractal ensemble of particles and in large disordered fractal systems. In the book: "*Turbulence,*

dynamics of the atmosphere and climate" (Collection of proceedings of the International conference "Turbulence, dynamics of the atmosphere and climate", dedicated to the centenary of the birth of Academician AM Obukhov (Moscow, 16-18.05.2018) Ed. By GS Golitsyn et al. Moscow, Fizmatkniga Publ., 2018, pp. 564-573.

42. Potapov AA. Is it possible to build a fractal radio system? *Review of Applied and Industrial Mathematics*, 2007, 14(4):742-744.
43. Potapov AA. My meeting with B. Mandelbrot. *Nelineinyi mir,* 2007, 5(6):402-404.
44. Potapov AA. New information technologies based on probabilistic texture and fractal features in radar detection of low-contrast targets. *Radio engineering and electronics,* 2003, 48(9):1101-1119.
45. Potapov AA. On the concept of fractal radio systems and fractal devices. *Nelineinyi mir*, 2007, 5(7-8):415-444.
46. Potapov AA. On the application of the Hurst exponent H in adaptive fractal information processing and the synthesis of new classes of fractal "H-signals". *Review of Applied and Industrial Mathematics*, 2008, 15(6):1121-1123.
47. Potapov AA. On strategic directions in the synthesis of new types of radar texture-fractal detectors of low-contrast objects with the allocation of their contours and localization of coordinates against the background of intense interference from the surface of the earth, sea and precipitation. *Proceedings of the IV All-Russian STC "RTI Systems VKO-2016", dedicated. To the 100th anniversary of NIIDAR and the 70th anniversary of the RTI named after V.I. acad. AL Mints* (Moscow, JSC "RTI named after Academician AL Mints", 02-03.06.2016). Moscow, Bauman MSTU Publ., 2017, p. 438-448.
48. Potapov AA. On fractal radiophysics and fractal radio electronics. *Sat. report anniversary. scientific and technical conf. "Innovations in radio information and telecommunication technologies", dedicated to the 60th anniversary of JSC "Radiotekhn. inst.im. acad. AL Mints "and the Faculty of Radio Electronics of Aircraft MAI* (Moscow, 24-26.10.2006). Moscow, Extra Print, 2006, Part 1, p. 66-84.
49. Potapov AA. About fractal radio systems, fractional operators, scaling, and more ... Chapter in the book: *Fractals and fractional operators*. With a preface acad. YuV Gulyaev and Corresponding Member RAS SA Nikitov. Kazan, "Fan" Acad. sciences RT Publ., 2010, p. 417-472.
50. Potapov AA. Fractal fluctuations of microwave radio waves in an absorbing medium and negative fractal dimension. *Review of Applied and Industrial Mathematics,* 2008, 15(6): 123-1124.
51. Potapov AA. Scattering of waves on a stochastic fractal surface. *Sat. scientific works for the 65th anniversary of the creation of the Kotelnikov IRE of RAS and the 110th anniversary of the birth. acad. VA Kotelnikov.* Ed. Corresponding Member RAS SA Nikitov. Moscow, Kotelnikov IRE of RAS Publ., 2018, p. 155-159.
52. Potapov AA Modern fractal radio systems and technologies (40 years of scientific development): the basics of fractal-scaling or scale-invariant radar. *Proc. of the V Int. scientific. conf. "Nonlocal boundary value problems and related problems of mathematical biology, computer science and physics", dedicated. 80th anniversary of AM Nakhushev* (Nalchik, KBR, Russia, 4-7.12.2018). Nalchik, IPMA KBSC RAS Publ., 2018, p. 165-166.
53. Potapov AA. Statistical approach to the description of images of textures of the earth's surface in the optical and radio bands. *Abstracts. report V ses. conf. "Mathematical Methods for Pattern Recognition* (MMRO-IV)" (Riga, 24-26.10.1989). Riga, MI-PKRRiS Publ., 1989, 4:150-151.
54. Potapov AA. Texture and fractal scaling methods for detecting, processing and recognizing weak radar signals and low-contrast images against a background of intense interference. *Aerospace Defense Bulletin*, 2018, 2(18):15-26.
55. Potapov AA. Textures, fractals, fractional operators and nonlinear dynamics methods in radiophysics and radar: 40 years of scientific research. *Sat. works XXV Int. STC "Radar, navigation, communication", dedicated. 160th anniversary of the birth. AS Popov* (Voronezh, 16-18.04.2019). Voronezh, Voronezh State University Publ., 2019, 4:214-242.
56. Potapov AA. Turbulence, fractals and waves. *Sat. thesis. report Int. conf. "Turbulence, dynamics of the atmosphere and climate", dedicated. centenary from the date of birth. acad. A.M. Obukhov* (05.05.1918-03.12.1989) (Moscow, 16-18.05.2018). Moscow, Fizmatkniga Publ., 2018, p. 211.
57. Potapov AA. *Fractals in radio physics and radar.* Moscow, Logos Publ., 2002, 664 p.
58. Potapov AA. *Fractals in Radiophysics and Radar: Sample Topology.* Ed. 2nd, rev. and add. Moscow, University book Publ., 2005, 848 p.
59. Potapov AA. Fractals and fractional operators in radio engineering, radar and multidimensional

signal processing. *Abstracts. report Int. NTK to the 100th anniversary of the birth. VA Kotelnikov* (Moscow, 21-23.10.2008). Moscow, MPEI Publ., 2008, p. 29-33.

60. Potapov AA. Fractals and negative capacitor. *News from int. Chinese-Russian Symposium "New Materials and Technologies"* (Hainan, China, 28.11-1.12.2017). Sat. tr. XXIV int. STC "Radar, navigation, communication" (Voronezh, 17-19.04.2018). Voronezh, Ed. "Scientific research. publications "(WELBORN LLC), 2018, 3:372-388.
61. Potapov AA. Fractals and Textures in Radiophysics and Radio Electronics: 40 Years of Scientific Research. *Sat. mater. XIV int. conf. "Zababakhin Scientific Readings"* (Snezhinsk, 18-22.03.2019). Snezhinsk, RFNC-VNIITF Publ., 2019, p. 105-107.
62. Potapov AA. Fractals and chaos as the basis of new breakthrough technologies in modern radio systems. Add. book: Kronover R. *Fractals and chaos in dynamical systems*. Per. from English; ed. TE Krenkel. Moscow, Technosphere Publ., 2006, p. 374-479.
63. Potapov AA. Fractals, scaling and fractional operators in modern physics and radio engineering. *Sat. annotations Int. conf. XIV Kharitonov thematic scientific readings "Powerful Pulsed Electrophysics", dedicated. 110th anniversary of the birth. acad. YB Khariton* (Sarov, 21-25.04.2014). Sarov, RFNC-VNIIEF Publ., 2014, p. 80-81.
64. Potapov AA. Fractal radio electronics: state and development trends. *Sat. scientific. Art. by mater. III All-Russia. scientific-practical conf. Avionics* (15-16.03.2018). Voronezh, VUNC VVS "VVA im. prof. NOT Zhukovsky and YA Gagarin" Publ., 2018, p. 267-272.
65. Potapov AA. Fractal electrodynamics. Numerical modeling of small fractal antenna devices and fractal 3D microstrip resonators for modern ultra-wideband or multiband radio engineering systems. *Radio Engineering and Electronics,* 2019, 64(7):629-665.
66. Potapov AA. Fractal-scaling or scale-invariant radar: discovery, justification and development paths. *Sat. scientific. Art. by mater. II All-Russia. scientific-practical conf. Avionics* (Voronezh, March 16-17, 2017). Voronezh, VUNTS VVS "VVA im. prof. NOT Zhukovsky and YA Gagarin" Publ., 2017, p. 143-152.
67. Potapov AA. Fractal scaling or scale invariant radar and fractal signal and image processing. *Sat. scientific. works for the 65th anniversary of the creation. Kotelnikov IRE of RAS and the 110th anniversary of the birth. acad. VA Kotelnikov.* Ed. Corresponding Member RAS SA Nikitov. Moscow, IRE RAS Publ., 2018, p. 99-104.
68. Potapov AA. Fractal and texture detectors of weak radar signals against a background of intense interference. Part I. Introduction to the principles of topological detectors. *Radio engineering,* 2019, 1:80-92.
69. Potapov AA. Fractal methods for studying fluctuations of signals and dynamical systems in a space of fractional dimension. Chapter in the book: "*Fluctuations and noises in complex systems of animate and inanimate nature.*" Ed. RM Yulmetyeva et al. Kazan, Min. Education and Science Rep. Tatarstan Publ., 2008, p. 257-310.
70. Potapov AA. Fractal models and methods in problems of nonlinear physics. *Abstracts. report int. Congr. "Nonlinear dynamic analysis-2007", dedicated. 150th anniversary of the birth. acad. AM Lyapunov* (St. Petersburg, June 4-8, 2007). SPb., SPGU Publ., 2007, p. 301.
71. Potapov AA. Fractal models and methods based on fractional operators and scaling in fundamental and applied problems of physics. *Proc. 2 int. "Mathematical physics and its applications"* (Samara, 29.08-04.09.2010). Ed. Corresponding Member RAS IV Volovich and Doctor of Physical and Mathematical Sciences, prof. YUN Radaeva. Samara, Book Publ., 2010, p. 266-268.
72. Potapov AA. Fractal models and methods based on scaling in fundamental and applied problems of modern physics. *Sat. scientific. tr. "Irreversible processes in nature and technology."* Ed. VS Gorelik and AN Morozov. Moscow, Bauman MSTU Publ., 2008, II:5-107.
73. Potapov AA. *Fractal method and fractal paradigm in modern natural science.* Voronezh, Scientific book Publ., 2012, 109 p.
74. Potapov AA, Bulavkin VV, German VA, Vyacheslavova OF. Investigation of the microrelief of the processed surfaces using the methods of fractal signatures. *ZhTF*, 2005, 75(5):28-45.
75. Potapov AA, German VA. Application of fractal methods for processing optical and radar images of the earth's surface. *Radio Engineering and Electronics*, 2000, 45(8):946-953.
76. Potapov AA, German VA. Fractal nonparametric radio signal detector. *Radio engineering,* 2006, 5:30-36.
77. Potapov AA, Gulyaev YuV, Nikitov SA, Pakhomov AA, German VA. The latest methods of image

processing. Ed. AA Potapova. Moscow, Fizmatlit Publ., 2008, 496 p. (RFBR grant No. 07-07-07005).
78. Potapov AA, Laktyunkin AV. The theory of wave scattering by a fractal anisotropic surface. *Nelineinyi mir*, 2008, 6(1):3-36.
79. Potapov AA, Potapov AA (Jr.), Potapov VA. Fractal capacitor, fractional operators and fractal impedances. *Nelineinyi mir,* 2006, 4(4-5):172-187.
80. Potapov AA, Chernykh VA. *Fractional calculus AV Letnikov in the physics of fractals*. Saarbrücken, LAMBERT Academic Publishing, 2012, 688 pp.
81. Potapov AA, German VA. The effects of deterministic chaos and a strange attractor in the radar of a dynamic system such as vegetation. *ZhTF Letters*, 2002, 28(14):19-25.
82. *Professor Alexander Alekseevich Potapov. Biobibliographic Index*. Ed. Academician Yu.V. Gulyaev. Moscow, CPU "Raduga" Publ., 2019, 256 p. (Approved by the Academic Council of the Kotelnikov IRE of RAS on December 26, 2018).
83. Potapov AA. Brief scientific biography. In the book: "*International Forum for Industrial Development of New Materials*" (Jining, China, 11-13.12.2019). Jining: Jining National High-tech Industrial Development Zone, 2019, p. 8 (Chinese, Japanese, Russian).
84. Potapov AA. Application of the principles of fractal-scaling or scale-invariant radar in SAR, UAV and MIMO systems. In the book: *Radar. Results of theoretical and experimental research. In 2 books. Book. 2. Ed. AB Blyakhmana*. Moscow, Radiotekhnika Publ., 2019, p. 15-39.
85. Saks S. *Teoriya integrala*. Moscow, IL Publ., 1949, 496 p.
86. Samko SG, Kilbas AA, Marichev OI. *Integrals and derivatives of fractional order and some of their applications*. Minsk, Sci. and techn. Publ., 1987, 688 p.
87. Stratonovich RL. *The principles of adaptive reception*. Moscow, Sov. radio Publ., 1973, 144 p.
88. Uchaikin VV. *Fractional Derivatives Method*. Ulyanovsk, Artichoke Publ., 2008, 512 p.
89. Feder E. *Fractals*. Moscow, Mir Publ., 1991, 262 p.
90. Fedorchuk. BB. *Foundations of dimension theory. Results of Science and Technology. Modern probl. mat. Fund. directions. General topology–1*. Moscow, VINITI Publ., 1988, 17:111-124.
91. Feller V. *Introduction to the theory of probability and its applications*. Moscow, Mir Publ., 1967, vols. 1,2.
92. *Fractals in physics*. Per. from English ed. YAG Sinai and IM Khalatnikov. Moscow, Mir Publ., 1988, 672 p.
93. Frisch W. *Turbulence*. Legacy of AN Kolmogorov. ML Blank (ed.). Moscow, Fazis Publ., 1998, 346 p.
94. Halmos P. *Measure theory*. Moscow, IL Publ., 1953, 292 p.
95. Hausdorff F. *Set theory*. Per. with it.; ed. and with add. PS Aleksandrov and AN Kolmogorov. Moscow-Leningrad, ONTI Publ., 1937, 304 p.
96. Schroeder M. *Fractals, chaos, power laws*. Moscow-Izhevsk, Research Center "Regular and Chaotic Dynamics" Publ., 2001, 528 p.
97. Shuster G. *Deterministic chaos: an introduction*. Per. from English ed. AV Gaponov-Grekhov and MI Rabinovich. Moscow, Mir Publ., 1988, 240 p.
98. Yaglom AM. *Correlation theory of stationary random functions*. Leningrad, Gidrometeoizdat Publ., 1981, 280 p.
99. Agalarov AM, Gadzhimuradov TA, Potapov AA, Rassadin AE. Edge States and Chiral Solitons in Topological Hall and Chern-Simons Fields. *Modeling and Analysis of Information Systems,* 2018, 25(1):133-139.
100. Alisultanov ZZ, Agalarov AM, Potapov AA, Ragimkhanov GB. Some Applications of Fractional Derivatives in Many-Particle Disordered Large Systems. Ch. 7 in: *Fractional Dynamics, Anomalous Transport and Plasma Science*. Ed. C. Skiadas. Switzerland, Springer Int. Publ., 2018, p. 125-154.
101. Anastassiou GA. *Fractional Differentiation Inequalities*. N.Y., Springer, 2009, 686 p.
102. *Applications of Fraction Calculus in Physics*. Ed. by R. Hilfer. Singapore, World Scientific Publishing Co., 2000, 472 p.
103. Caputo M. *Elasticita e Dissipacione*. Bologna, Zanichelli Publ., 1969.
104. Caputo M. Linear models of dissipation whose Q is almost frequency independent. II. *Geophys. J.R. Astr. Soc.*, 1967, 13:529-539.
105. Caratheodory C. *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*. Leipzig-Berlin, Teubner, 1935, 407 s. (English transl.: *Calculus of Variations and Partial Differential Equations of the First Order*. N.Y., Chelsea Publishing Company, 1965, 654 p.
106. Edgar GA. (ed.) *Classics on Fractal*. N.Y., Addison-Wesley, 1993, 366 p.
107. Danping Pan, Tianhua Feng, Wei Zhang, and Alexander A. Potapov. Unidirectional light scattering by electric dipoles induced in plasmonic nanoparticles. *Opt. Lett.,* 2019, 44(11):2943-2946.
108. Foukzon J, Potapov AA, Podosenov SA. Hausdorff-Colombeau measure and axiomatic quantum field theory in spacetime with negative B. Mandelbrot dimensions. http://arxiv.org/abs/1004.0451, 5 Feb. 2011, 206 c.

109. German VA, Potapov AA, Sykhonin EV. Fractal Characteristics of Radio Thermal Radiation of a Different Layer of Atmosphere in a Range of Millimeter Waves. *Proc. PIERS 2009 "Progress in Electromagnetics Research Symp"* (18-21.08.2009, Moscow, Russia). Cambridge, MA, Electromagnetics Academy, 2009, pp. 1813-1817.

110. Hata M. Fractals in Mathematics. Pattern and Waves: Qualitative Analysis of Nonlinear Differential Equations. *Studies in Mathematics and its Applications, V. 18.*/Ed. by T. Nishida, M. Mimura, H. Fujii. Tokyo, Kinokuniya Comp. Ltd., 1986, pp. 259-278.

111. Kiryakova V. *Generalized Fractional Calculus and Applications.* N.Y., Wiley & Sons, 1994, 360 p.

112. Kolwankar KM, Gangal AD. Fractional Differentiability of Nowhere Differentiable Functions and Dimensions. *Chaos*, 1996, 6(1):505-513.

113. Laktyunkin Alexander, Potapov Alexander A. The Hurst Exponent Application in the Fractal Analysis of the Russian Stock Market. In: *Advances in Artificial Systems for Medicine and Education II.* Ed: Z. Hu, S. Petoukhov, M. He (Part of the *Advances in Intelligent Systems and Computing book series – AISC, V. 902*). Cham, Switzerland, Springer Int. Publ., 2018, pp. 459-471.

114. Mainardi F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models.* London, Imperial College Press, 2010, 368 p.

115. Mandelbrot B. *The Fractals Geometry of Nature.* N.Y., Freeman, 1982, 468 p.

116. Mathai AM, Saxena RK. *The H-Function with Applications in Statistics and Other Disciplines.* New Delhi, Wiley Eastern Limited, 1978, 192 p.

117. McBride AC. *Fractional Calculus and Integral Transforms of Generalized Functions.* San Francisco, Pitman Press, 1979, 179 p.

118. Metzler R, Klafter J. The random walk's guide to anomalous di!usion: a fractional dynamics approach. *Physics Reports,* 2000, 339:1-77.

119. Miller KS, Ross B. *Introduction to the Fractional Calculus and Fractional Differential Equations.* N.Y., Wiley, 1993, 384 p.

120. Nishimoto K. *Fractional Calculus. V. 1-5.* Koriyama (Japan), Descartes Press Co., 1984, v.1.-195 p.; 1987, v.2 – 189 p.; 1989, v.3 – 202 p.; 1991, v.4 – 158 p.; 1996, v.5 – 193 p.

121. Oldham KB, Spanier J. *The Fractional Calculus.* N.Y., Academic Press, 1974, 234 c.

122. Podlubny I. *Fractional Differential Equations.* N.Y., Academic Press, 1999, 368 p.

123. Podosenov SA, Foukzon J, Potapov AA. A Study of the Motion of a Relativistic Continuous Medium. *Gravitation and Cosmology,* 2010, 16(4):307-312.

124. Potapov AA, German VA. Detection of Artificial Objects with Fractal Signatures. *Pattern Recognition and Image Analysis,* 1998, 8(2):226-229.

125. Potapov AA. Fractal and topological sustainable methods of overcoming expected uncertainty in the radiolocation of low-contrast targets and in the processing of weak multi-dimensional signals on the background of high-intensity noise: A new direction in the statistical decision theory. *IOP Conf. Ser.: Journal of Physics*, 2017, 918(012015):19.

126. Potapov AA. The Textures, Fractal, Scaling Effects and Fractional Operators as a Basis of New Methods of Information Processing and Fractal Radio Systems Designing. *Proc. SPIE*, 2009, 7374:73740E-1–14.

127. Potapov AA. On the Issues of Fractal Radio Electronics: Part 1. Processing of Multi-dimensional Signals, Radiolocation, Nanotechnology, Radio Engineering Elements and Sensors. *Eurasian Physical Technical Journal,* 2018, 15(2(30)):5-15.

128. Potapov AA, Potapov Alexey A, Potapov VA. Fractal Radioelement's, Devices and Fractal Systems for Radar and Telecommunications. *Proc. 14th Sino-Russia Symposium on Advanced Materials and Technologies* (Sanya, Hainan Province, China, 28.11-01.12.2017). Ed. Mingxing Jia. Beijing, Metallurgical Industry Press (China), 2017, pp. 499-506.

129. Potapov AA. On the Issues of Fractal Radio Electronics: Part 2. Distribution and Scat-tering of Radio Waves, Radio Heat Effects, New Models, Large Fractal Systems. *Eurasian Physical Technical Journal*, 2018, 15(2(30)):16-23.

130. Potapov Alexander A. Postulate "The Topology Maximum at the Energy Minimum" for Textural and Fractal-and-Scaling Processing of Multidimensional Super Weak Signals against a Background of Noises. Глава 3 в кн.: *Nonlinearity: Problems, Solutions and Applications*. Ed. LA Uvarova, AB Nadykto, and AV Latyshev. N.Y., Nova Science Publ., 2017, 2:35-94.

131. Potapov Alexander A. Chaos Theory, Fractals and Scaling in the Radar: A Look from 2015. Глава 12 в кн.: *The Foundations of Chaos Revisited: From Poincaré to Recent Advancements*. Ed. C. Skiadas. Switzerland, Basel, Springer Int. Publ., 2016, pp. 195-218.

132. Potapov Alexander A. On the Indicatrixes of Waves Scattering from the Random Fractal Anisotropic Surface. Глава 9 в кн.: *Fractal Analysis - Applications in Physics, Engineering and Technology*. Ed. Fernando Brambila. Rijeka, InTech, 2017, pp. 187-248.
133. Potapov Alexander A, Pakhomov Andrey A, Grachev Vladimir I. Development of Methods for Solving Ill-Posed Inverse Problems in Processing Multidimensional Signals in Problems of Artificial Intelligence, Radiolocation and Medical Diagnostics. In: *Advances in Artificial Systems for Medicine and Education II*. Ed: Z. Hu, S. Petoukhov, M. He (Part of the Advances in Intelligent Systems and Computing book series–AISC, V. 902). Cham, Switzerland, Springer Int. Publ., 2018, pp. 57-67.
134. Rogers CA. *Hausdorff Measures*. London, Cambridge University Press, 1970, 179 p.
135. Rubin B. *Fractional Integrals and Potentials*. Harlow, Longman, 1996, 409 p.
136. Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives: Theory and Applications*. N.Y., Gordon and Breach, 1993, 688 p.
137. Potapov Alexander A, Wu Hao, Xiong Shan. *Fractality of Wave Fields and Processes in Radar and Control*. Guangzhou, South China University of Technology Press (First edition: November 2020), 2020, 280 p. – ISBN 978-7-5623-6276-0.
138. Potapov AA, Kuznetsov VA, Pototsky AN. A new class of topological texture-multifractal features and their application for the processing of radar and optical low-contrast images. Radio Engineering and Electronics, 2021, 66 (5); 457-467. DOI: 10.31857 / S0033849421050107.
139. Kuznetsov VA, Potapov AA, Alikulov EA. Method of fractal complexing of multifrequency radar images. Patent 2746038 RF. MPK8 N 01 G 19/18 (2006.01). (The priority of the invention is 09/05/2020, the date of state registration is 04/06/2021).
140. Pan Danping, Wan Lei, Potapov Alexander A, and Feng Tianhua. Performing Spatial Differentiation and Edge Detection with Dielectric metasurfaces. *QELS_Fundamental Science "OSA Technical Digest Conf. on Lasers and Electro-Optics* (CLEO) (San Jose, California, USA, 10-15 May 2020)". Washington: Optical Society of America, 2020. Paper FW4B.2.pdf.- 2 p. (From the session "Inverse Design and Computation (FW4B)").
141. Lei Wan, Danping Pan, Shuaifeng Yang, Wei Zhang, Potapov Alexander A, Xia Wu, Weiping Liu, Tianhua Feng, and Zhaohui Li. Optical analog computing of spatial differentiation and edge detection with dielectric metasurfaces. *Opt. Lett.*, 2020,45(7):2070-2073.https://www.osapublishing.org/ol/abstract.cfm?URI=ol-45-7-2070.
142. Tianhua Feng, Potapov Alexander A., Zixian Liang, and Yi Xu. Huygens Metasurfaces Based on Congener Dipole Excitations. *Physical Review Applied,* 2020, 13(021002):1-6. DOI: 10.1103/PhysRevApplied.13.021002.
143. Tianhua Feng, Shuaifeng Yang, Ning Lai, Weilian Chen, Danping Pan, Wei Zhang, Potapov Alexander A, Zixian Liang, and Yi Xu. Manipulating light scattering by nanoparticles with magnetoelectric coupling. *Phys. Rev. B.*, 2020, 102(205428): 7 p. – (Published 30 November 2020). DOI: 10.1103/PhysRevB.102.205428.
144. Lei Wan, Danping Pan, Tianhua Feng, Weiping Liu, Potapov A.A. A review of dielectric optical metasurfaces for spatial differentiation and edge detection. *Frontiers of Optoelectronics*, 2021, 14 p. DOI: 10.1007/s12200-021-1124-5.
145. Danping Pan, Lei Wan, Min Ouyang, Wei Zhang, Potapov Alexander, Weiping Liu, Zixian Liang, Tianhua Feng, Zhaohui Li. Laplace metasurfaces for optical analog computing based on quasi-bound states in the continuum. *ACS Photonics,* 2021, 8(3)