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Improving the accuracy of measurement time intervals of radio reception in the framework of recursive multi-stage Bayesian estimates

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Abstract. The conditions of increasing the accuracy of digital measurement of repeating signal parameters (increasing the resolution) in measuring radio reception systems by reducing sampling and quantization errors, with a random nature of the change in the measurement conditions (experiment conditions) are analyzed. The method of reducing the errors of digital measurement during random modulation of the phase of the quantizing sequence and conducting multi-stage measurements using recursive optimal estimates from the theory of statistical estimation is analyzed in detail. By analogy with adaptive digital filters, a hypothesis is put forward about the possibility of increasing instrumental accuracy (reducing instrumental errors) in measuring repeated time intervals with random modulation of the initial phase of quantizing pulses and using feedback in the experiment. Thus, the fundamental possibility of creating the fundamental basis of ultra-precise measurements is proved. Similarly, an increase in image sharpness along the X, Y axes is realized, for example, the aperture of the line of receiving sensors and the use of rough ("Boolean") statistics of the input signal. The approach based on the recursion of measurements due to changes in the conditions of the experiment using the statistical test method (Monte Carlo method) is considered. To clarify (interpolate) the position of the end of the measured segment inside the resolution element and thus achieve the effect of "instrumental overresolution" of multiple measurements.

Keywords: chaotization of measurements (randomization), chaotic modulation of the phase, rough counting, measurement recursion, instrumental accuracy, discretization and quantization effects, feedback, stochastic linearization

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1. INTRODUCTION

In a number of problems of a radiolocation and the radio measuring technics it is not required derivings of exact instant references of time intervals, therefore measuring process can be organised not on one, and on a series (pack) n of impulses. In that case when the modification of the measured parametre can be neglected, an instrumental error reduce at the expense of randomization of algorithms of measuring at usage of statistical handling of "rough" current references of time intervals. As a result of it high-speed performance of equipment, in particular metering circuits (measuring of time intervals by the specified mode makes a subject of engineering application of a known Monte-Carlo method in calculus mathematics [1, 2]) decreases. Enough the close engineering problem (to the mentioned measurer). The image processing problem is. As in that, and other case, it is primary (at measuring)

it is used the method of the direct account, however remains some error of step-type behaviour which cannot be subjected a method of the direct account any more since it becomes less prices of division of a measuring dial of the tool.

2. PROBLEM STATEMENT

Immediate usage of a Monte-Carlo method (method of statistical trials of SI) does not allow to gain, unfortunately, enough effective valuation because of low speed of convergence of evaluations. We will illustrate it on an example of measuring of an iterated time interval. Measuring (estimation) by a SI method is led [3] by count of number m of coincidence of two independent sequences. The first sequence consists of short quantizing impulses (scores) with known reference phase τ_0 , second – from n times of an iterated measured time interval duration τ . The second sequence has the phase of a recurring modulated on the casual law T . Modulation of phase of a recurring of measured impulses can be created artificially, for example by insert in a chain of start radar station (RLS) of the wobbler of repetition frequency.

If the moments of occurrence of measured intervals are independent and are uniformly proportioned within an

interval τ_0 "rough" references of time intervals on separate trials will differ on magnitude $\mu_i \tau_0$ [3], where

$$\mu_i = \begin{cases} 1, & \text{with probability } p = R\{\tau / \tau_0\}, \\ 0, & \text{with probability } q = 1 - p, \end{cases}$$

$R\{\tau / \tau_0\}$ function of a fractional share of the ratio τ / τ_0 .

The statistics of coincidence $m = \sum_{i=1}^n \mu_i$ in n trials is proportioned under the binomial law:

$$W\left(\frac{m}{n} / p\right) = c_n^m p^m q^{n-m}, \quad (1)$$

c_n^m – number of combinations of n by m .

The probability p can be estimated on frequency of coincidence $p^* = m/n$ therefore a τ^* time interval estimate τ

$$\tau^* = \tau_0(E\{\tau / \tau_0\} + p^*) = \tau_0(E\{\tau / \tau_0 + m/n\}). \quad (2)$$

$E\{\tau / \tau_0\}$ function of an integer part of the ratio τ / τ_0 .

Medial on all m quadrate of an error of estimation (2) taking into account expression (1) we will gain in an aspect

$$\begin{aligned} M\{(\tau^* - \tau)^2 / p\} &= \tau_0^2 M\{(p^* - p)^2 / p\} = \\ &= \tau_0^2 D_0 = \tau_0^2 \sigma^2, \end{aligned} \quad (3)$$

where $\sigma = \sqrt{D_0} = \sqrt{M\{(p^* - p)^2 / p\}} = \sqrt{pq/n}$ mean squared error of measuring of probability p on frequency m/n , $M\{\cdot\}$ expectation operation.

From the equation (3) we will discover a mean squared error of estimation σ_τ оценки интервала of an interval τ

$$\sigma_\tau = \sqrt{M\{(\tau^* - \tau)^2 / p\}} = \tau_0 \sigma. \quad (4)$$

From the formula (4) it is visible that the error σ_τ of measuring of an interval τ (at admissible on high-speed performance of applied equipment a quantization interval τ_0) is completely spotted by an error of measuring σ of probability p which depending on number n of processed impulses decreases rather slowly and has the order $n^{1/2}$.

It is easy to show that the $p^* = m/n$ probability estimate p is a maximum likelihood estimate converting into a maxima function of verisimilitude (1), in a sense understood as function from a variable p .

Lowering reviewing of obvious problems on a competence and efficiency of a maximum likelihood estimate, it is necessary to score that the lower bound of its error gained from an inequality of Rao-Kramera [4], is spotted also by expression (4).

In paper the answer to a problem is given: whether it is possible to raise in addition a measurement accuracy of probability p if field of maximum likelihood estimates to dilate more blanket nonrecursive and recursive Bayes estimates. In a sense this problem can be considered as a problem of "an instrumental supersolution».

3. BAYES ESTIMATES OF PROBABILITY OF A SOFTWARE TO FREQUENCY AND MAXIMUM LIKELIHOOD ESTIMATES

If the aprioristic cumulative distribution function to designate through $W(p)$, posteriori – $w\left(p/\frac{m}{n}\right)$ verisimilitude function – $w\left(\frac{m}{n}/p\right)$ at a quadratic loss function the Bayes estimate p_{opt}^* of probability p will be noted as follows:

$$p_{opt}^* = \int_0^1 p W\left(p/\frac{m}{n}\right) dp = \int_0^1 \frac{p W\left(\frac{m}{n}/p\right)}{G\left(\frac{m}{n}\right)} W(p) dp. \quad (5)$$

Here $G\left(\frac{m}{n}\right) = \int_0^1 w\left(\frac{m}{n}/p\right) W(p) dp$ normalising coefficient. In the formula (5) it is possible to use floppy enough aprioristic probability density $W(p)$ in the form of [5] β allocations:

$$W(p) = \frac{\tilde{A}(v_1 + v_2)}{\tilde{A}(v_1)\tilde{A}(v_2)} p^{v_1-1} (1-p)^{v_2-1}, \quad p \in [0, 1], \quad (6)$$

where $\Gamma(v)$ – γ function.

Selection of parametres v_1 also v_2 it is possible to gain a major set of allocations (including uniform at $v_1 = v_2 = 1$), practical problems corresponding to a wide range. Association on p the conditional medial quadrature of an error $M\{(p^* - p)^2/p\}$ for a maximum likelihood estimate and a bayesian estimate is reduced in a **Fig. 1**.

By means of formulas (1), (5), (6) we will gain a bayesian (optimal) estimate

$$p_{opt}^* = (m + v_1) / (n + v_1 + v_2). \quad (7)$$

Fig. 1. Medial quadrature of an error for a maximum likelihood estimate (a shaped line) and a bayesian estimation (solid line) for $n = 16$ and $n = 64$.

Medial quadrature of an error of measuring of probability p for the uniform aprioristic law $W(p) = 1$ ($v_1 = v_2 = 1$):

$$M\{(p_{opt}^* - p)^2 / p\} = \frac{np(1-p) + (1-2p)^2}{(n+2)^2}. \quad (8)$$

Comparing medial quadrates of errors of Bayes and maximum probable p_m^* estimates, it is possible to discover cross points (drawing) of curves $M\{(p_{opt}^* - p)^2 / p\} = \frac{np(1-p) + (1-2p)^2}{(n+2)^2}$. Differently, in a gamut $p < 0.15$ and $p > 0.85$ it is necessary to use a maximum likelihood estimate, in a remaining gamut the p Bayes estimate is better. However advantages of a maximum likelihood estimate in the specified gamuts are reached only when the measurand lies in them, i.e. the aprioristic information concerning magnitude of measured probability is necessary.

For a raise of a measurement accuracy or cutting of number of trials when all aprioristic informations are settled, it

is necessary to discover new reserves. Step-by-step procedure of measuring of probability when observed datas at the previous stages are considered as the aprioristic informations for the subsequent stages, does not give additional scorings. Really, if by results of measuring at each stage to take a maximum likelihood estimate

$$p^*(m_i) = m_i / n_i, \quad i = 1, 2, 3, \dots, l,$$

where l number of stages of measuring; n_i sizes of each stage on observed datas after l stages it is possible to generate a resultant an estimate:

$$p_\Sigma^* = \sum_{i=1}^l \eta_i p^*(m_i) \quad (9)$$

(η_i – unknown weighting coefficients), and

$$\sum_{i=1}^l \eta_i = 1. \quad (10)$$

Taking into account limiting (10), optimum weighting coefficients η_{iOOP} , $i = 1, 2, \dots, l$ error of estimations (9) minimising the conventional medial quadrate, are gained in an aspect

$$\eta_{iopt} = n_i / \sum_{i=1}^l n_i. \quad (11)$$

Substituting the equation (11) in (9), we will have

$$p_\Sigma^* = \sum_{i=1}^l m_i / \sum_{i=1}^l n_i = m / n.$$

Thus, the total estimate p_Σ^* coincides with a maximum likelihood estimate одноэтапной procedures for $n = \sum_{i=1}^l n_i$.

4. FEEDBACK COUPLING IN THE TIME INTERVAL METER THE METHOD OF STATISTICAL TRIALS

The previous outcomes have spotted limits of a method of statistical trials at the expense of use of the aprioristic informations. In that case when the aprioristic informations are settled completely, it is necessary to discover essentially new reserve of a raise of accuracy of a method of statistical trials. The supposition about a possibility of application of feedback coupling in estimation procedure is with that end in view made. We will divide all volume of trials n into two and more stages. The first stage in volume n_1 represents the classical plan of independent trials of Bernulli. At development of an estimate p_1^* of initial probability $p = R\{\tau/\tau_0\}$ at the first stage it is necessary to use all available aprioristic informations. Differently, the estimate p_1^* is optimum Bayes. Further on an estimate p_1^* the correcting reference component δ which is added to an initial unknown quantity of measured probability is made p . Thus, at the second stage in volume n_2 measuring of new magnitude is spent

$$p_2 = p + \delta. \quad (12)$$

The estimate p_2^* by results of the second stage depends on observed datas at the first stage. Following the results of two stages the resultant an

estimate $p_{1,2}^* = p_1^* \eta_1 + (p_2^* - \delta) \eta_2$, where η_1, η_2 weighting coefficients is made, $\eta_1 + \eta_2 = 1$.

It is necessary to note that the assumption about a feedback coupling possibility at measuring is not always realizable. However measuring of time intervals by a method of statistical trials - an ideal case of embodying of offered idea. For introduction of the reference component, obviously, it is possible to apply a lag line with switching of taps. The reference component δ is equivalent to a delay of back edge of a measured interval at the second stage for a while $\delta \tau_0$. In some measure related idea is the urn plan of Polya [5]. However in the plan of Polya magnitude of the correcting component (number of full-spheres of certain colour) is constant, its sign (colour of added full-spheres) changes only. The prospective correcting component δ is function of the previous measurements $\delta = \delta(p_1^*)$.

As a matter of fact developed idea two-staged procedures the estimate after the first stage p_1^* is used for making of the best requirements of measuring of probability at the second stage. The previous outcomes testify that total measuring $p + \delta = 1$ as at $p = 0$ or $p = 1$ the conventional medial quadrate of an error the least will be the best requirement of measuring.

Thus, the algorithm of the correcting reference component is extremely simple:

$$\delta(p_1^*) = 1 - p_1^*. \quad (13)$$

Let's note expression for medial quadrate of an error of a resultant of an estimate $p_{1,2}^*$

$$\begin{aligned} M_{1,2} \{ (p_{1,2}^* - p)^2 \} &= \int_0^1 \left\{ \sum_{m_1=0}^{n_1} \left[\sum_{m_2=0}^{2n_2} (p_1^* + \eta_2(p_2^* - 1) - \right. \right. \\ &\quad \left. \left. - p)^2 W\left(\frac{m_2}{n_2} / m_1, p\right) \right] \times W\left(\frac{m_1}{n_1} / p\right) \right\} W(p) dp = \\ &= \int_0^1 \left\{ \sum_{m_2=0}^{n_1} \left[(p_1^* - p)^2 - 2\eta_2(p_1^* - p) \sum_{m_2=0}^{2n_2} (p_2^* - 1) \times \right. \right. \\ &\quad \left. \left. \times W\left(\frac{m_2}{n_2} / m_1, p\right) + \right. \right. \\ &\quad \left. \left. + \eta_2^2 \sum_{m_2=0}^{2n_2} (p_2^* - 1)^2 W\left(\frac{m_2}{n_2} / m_1, p\right) \right] W\left(\frac{m_1}{n_1} / p\right) \right\} W(p) dp = \\ &= D_1 - 2\eta_2 D_1 + \eta_2^2 D_2, \end{aligned} \quad (14)$$

where

$$D_1 = \int_0^1 \left[\sum_{m_1=0}^{n_1} (p_1^* - p)^2 W\left(\frac{m_1}{n_1} / p\right) \right] W(p) dp,$$

Optimising (14) on η_2 we will gain

$$\begin{aligned} M_{1,2opt} \{ (p_{1,2}^* - p)^2 \} &= \\ &= (D_1 - 2\eta_2 D_1 + \eta_2^2 D_2) / \eta_2 - \eta_{2opt} = \\ &= \frac{D_1}{D_2} (D_2 - D_1). \end{aligned} \quad (15)$$

Comparing $M_{1,2opt}$ with D_0 under condition of identical total amount of trials $n = n_1 + n_2$ for a case small $n = 2, 3, 4$ (see the **Table**), it is possible to draw following deductions:

1. The partition reduces medial quadrate of an error of measuring by stages in comparison with one-staged procedure at identical total volume of trials

Table
Calculation of the relative scoring B in percentage
% for various relations between stages n_1 and n_2 for
двухэтапной procedures.

Designation	"1+1"	"2+1"	"1+2"	"2+2"
n_1	1	2	1	2
n_2	1	1	2	2
$\gamma_{1,2opt}$	9/32	32/129	18/41	64/161
D_2	16/31	43/256	123/972	161/1536
D_1	1/18	1/24	1/18	1/24
D_0	1/24	1/30	1/30	1/36
$M_{1,2opt}$	23/576	27/3096	23/738	94/3864
$B = (D_0 - M_{1,2opt})/D_0$	4	5.9	6.5	9.5

2. With growth the n medial quadrate of an error at двухэтапной to procedure impinges faster, than at одноэтапной so the scoring grows.
3. The scoring depends on a relation of volumes between stages.

The analysis of medial quadrate (15) for a case of the major $n \geq 1$ is hampered owing to sectionally continuous exposition D_2 on the statistican m . However the analysis somewhat becomes simpler if to consider normalisation of function of probability $W(p_2/p)$ with growth n_1 . Outcomes of the basic calculations are tabulated.

From formulas (12), (13) and (7) it is visible that probability function $W(p_2/p)$ is actually binomial. Hence, with growth the n_1 asymptotics of the Moavre-Laplace, therefore is valid

$$W(p_2 / p) \cong \frac{1}{\sqrt{2\pi}\sigma_{p_2}} \exp\left\{\frac{(p_2 - a)}{2\sigma_{p_2}^2}\right\},$$

where a $\sigma_{p_2}^2$ variance of a frequency function an $W(p_2/p)$, a - average (on all m_1) value p_2 .

Considering conversions (12), (13) and (7), it is possible to show that

$$\sigma_{p_2}^2 = n_1 p(1 - p) / (n_1 + 2)^2,$$

$$a = (n_1 + 1 + 2p) / (n_1 + 2).$$

As a result of an average on m_2 , m_1 , p expressions D_1 and D_2 from (15) taking into account approximations

$$\{\Phi(a / \sigma_{p_2}) - \Phi(a / \sigma_{p_2} - 2)\} \cong 2,$$

$$\exp\{-(1 / 2\sigma_{p_2}^2)(2 - a)^2\} \cong 0,$$

где $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_0^u e^{-t^2/2} dt$ integral of Laplace, we will gain

$$D_2 = \frac{1}{3(n_1 + 2)^2} + \frac{n_1(n_2 - 1)}{n_2(n_1 + 2)} \frac{1}{6} + \frac{1}{4} \sqrt{\frac{\pi}{2}} \frac{1}{n_2 \sqrt{n_1}},$$

$$D_1 = 1/6(n_1 - 2).$$

As $n_1, n_2 \gg 1$ that

$$D_2 \cong 1/6n_1 + (1/4)\sqrt{\pi/2}(1/n_2\sqrt{n_1}),$$

$$D_1 \cong 1/6n_1.$$

Expression for medial quadrate of an error at двухэтапной to procedure looks like

$$M_{1,2opt} \cong D_2 - D_1 = (1/4)\sqrt{\pi/2}(1/n_2\sqrt{n_1}).$$

Examination $M_{1,2opt}$ on an extreme (minimum) at the fixed $n = n_1 + n_2$ gives a following optimum relation between stages: $n_2 = 2n_1$. Then

$$M_{1,2opt} \cong 3\sqrt{3\pi} / (8\sqrt{2}n\sqrt{n}) = 3^{3/2} \pi^{1/2} / (2^{7/2} n^{3/2});$$

$$\sigma_{1,2opt} = \sqrt{M_{1,2opt}} \cong 3^{3/4} \pi^{1/4} / (2^{7/4} n^{3/4}).$$

5. CONCLUSION

Modulation of phase of a recurring of measured impulses under the casual law at measuring of iterated time intervals by

a Monte-Carlo method allows to raise at handling of impulses a measurement accuracy in $\sqrt{n}$ time.

The raise of a measurement accuracy within the limits of nonrecursive (одноэтапных and многоэтапных) Bayes estimates is impossible without usage of the aprioristic informations concerning a measured interval.

The supposition about a feedback coupling possibility in a Monte-Carlo method is made and proved that an additional raise of a measurement accuracy of time intervals probably within the limits of the recursive Bayes estimates.

Application of recursive (2-etapnyh) Bayes estimates of measuring of time intervals gives the chance to increment a measurement accuracy in $n^{1/2} - n^{3/4}$ time.

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